LINEAR TRANSFORMATION

Definition

Let V and W be vector spaces over $F. A$ function $T: V \to W$ is called a linear transformation if for all $x, y \in V$ and $\alpha \in F$,

(a) $T(x+y) = T(x) + T(y)$

(b) $T(\alpha x) = \alpha T(x)$

Properties of linear transformation

1.If T is the linear, then $T(0) = 0$

Proof

$$T(0) = T(0+0)$$

$$T(0) = T(0) + T(0)$$

$\therefore T(0)$ is zero element of W.

Which implies,

$T(0) = 0$

2.T is linear if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$; for all $x, y \in$ V and $\alpha, \beta \in F$.

Proof

Assume T is linear.

$$T(\alpha x + \beta y) = T(\alpha x) + T(\beta y)$$

$$= \alpha T(x) + \beta T(y)$$

Conversely.

Assume $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ ... (1)

Put $\alpha = 1, \beta = 1$ in (1). Then

$T(x+y) = T(x) + T(y)$

Put $y = 0$ in (1). Then

$$T(\alpha x + 0) = \alpha T(x) + T(0)$$

$= \alpha T(x) + 0$

$= \alpha T(y)$

$\therefore T$ is linear.

3. If T is linear, then $T(x-y) = T(x) - T(y)$; for all $x, y \in V$ Given T is linear

$$T(x-y) = T(x + (-y))$$

$$= T(x) + T(-y)$$

$$= T(x) - T(y)$$

Example 1. $T: R^2 \to R^2$ is defined by $T(a_1, a_2) = (2a_1 + a_2, a_1)$. Verif whether T is a linear transformation

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2)$ and $y = (b_1, b_2)$

$x + y = (a_1 + b_1, a_2 + b_2)$

Given

$T(a_1, a_2) = (2a_1 + a_2, a_1)$

To prove T is linear, we have to prove

(i) $T(x + y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(y)$

Proof:

$f(x) = T(a_1, a_2)$

$=(2a_1 + a_2, a_1)$

$T'(y)= T(b_1, b_2)$

$=(2b_1 + b_2, b_1)$

(i) $r(x+y)= T(a_1+b_1, a_2+b_2))$

$= (2(a_1+b_1)+a_2+b_2, a_1+b_1)$

$= (2a_1+2b_1+a_2+b_2, a_1+b_1)$

$=(2a_1+a_2, a_1)+(2b_1+b_2, b_1)$

$= T(a_1, a_2)+T(b_1, b_2)$

$= T(x)+T(y)$

(ii) $T(\alpha x)= T(\alpha a_1, \alpha a_2)$

$=(2\alpha a_1+\alpha a_2, \alpha a_1)$

$= \alpha(2a_1+a_2, a_1)$

$= \alpha T(a_1, a_2) = \alpha T(x)$

$\therefore T: R^2 \to R^2$ is a linear transformation,

Example 2) $T: V_2(R) \to V_2(R)$ is defined by $T(a_1, a_2) = (3a_1 + 2a + 4a_2\ )$. Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2)$ and $y = (b_1, b_2)$

$x + y = (a_1 + b_1, a_2 + b_2)$

Given

$T(a_1, a_2) = (3a_1 + 2a_2, 3a_1 - 4a_2)$

To prove T is linear, we have to prove

(i) $T(x+y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(y)$

Proof:

$T(x) = T(a_1, a_2)$

$= (3a_1 + 2a_2, 3a_1 - 4a_2)$

$T(y) = T(b_1, b_2)$

$= (3b_1 + 2b_2, 3b_1 - 4b_2)$

(i) $T(x + y) = T(a_1 + b_1, a_2 + b_2))$

$= \big(3(a_1 + b_1) + 2(a_2 + b_2), 3(a_1 + b_1) - 4(a_2 + b_2)\big)$

$= (3a_1 + 3b_1 + 2a_2 + 2b_2, 3a_1 + 3b_1 - 4a_2 - 4b_2)$

$= (3a_1 + 2a_2 + 3b_1 + 2b_2, 3a_1 - 4a_2 + 3b_1 - 4b_2)$

$= (3a_1 + 2a_2, 3a_1 - 4a_2) + (3b_1 + 2b_2, 3b_1 - 4b_2)$

$= T(a_1, a_2) + T(b_1, b_2)$

$= T(x) + T(y)$

(ii) $T(\alpha x)= T(\alpha a_1, \alpha a_2)$

$=\big(3(\alpha a_1) + 2(\alpha a_2), 3(\alpha a_1) - 4(\alpha a_2)\big)$

$=\big(\alpha(3a_1 + 2a_2), \alpha(3a_1 - 4a_2)\big)$

$= \alpha\big((3a_1 + 2a_2), \alpha(3a_1 - 4a_2)\big)$

$= \alpha T(a_1, a_2) = \alpha T(x)$

$T: V_2(R) \to V_2(R)$ is a linear transformation.

Exmple 3. Define $T: R^3 \to R^3$ by $T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)$.

Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$

$x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Given

$T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)$

To prove T is linear, we have to prove

(i) $T(x+y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(x)$

Proof

$T(x) = T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)T(y) = T(b_1, b_2, b_3)$

$= (2b_1 + b_2, b_2 - b_3, 2b_2 + 4b_3)$

(i) $T(x+y) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$= \big(2(a_1 + b_1) + (a_2 + b_2), (a_2 + b_2) - (a_3 + b_3), 2(a_2 + b_2) + 4(a_3 + b_3)\big)$

$= (2a_1 + 2b_1 + a_2 + b_2, a_2 + b_2 - a_3 - b_3, 2a_2 + 2b_2 + 4a_3 + 4b_3)$

$= (2a_1 + a_2 + 2b_1 + b_2, a_2 - a_3 + b_2 - b_3, 2a_2 + 4a_3 + 2b_2 + 4b_3)$

$= (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3) + (2b_1 + b_2, b_2 - b_3, 2b_2 + 4b_3)$

$= T(a_1, a_2, a_3) + T(b_1, b_2, b_3)$

$= T(x) + T(y)$

$(ii) T(\alpha x) = T(\alpha a_1, \alpha a_2, \alpha a_3)$

$=(2\alpha a_1 + \alpha a_2, \alpha a_2 - \alpha a_3, 2\alpha a_2 + 4\alpha a_3)$

$=\big(\pi(2a_1 + a_2), \alpha(a_2 - a_3), \alpha(2a_2 + 4a_3)\big)$

$= \alpha(2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)$

$= \alpha T(a_1, a_2, a_3) = \alpha T(x)$

$\therefore T$ is a linear transformation and hence a linear map on R^3.

Example 4. Define mapping $T: V_3(F) \to V_2(F)$ by $T(a_1, a_2, a_3) = (a_2, a_3)$. Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2, a_3), y = (b_1, b_2, b_3)$

$x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Given

$T(a_1, a_2, a_3) = (a_2; a_3)$

To prove T is linear, we have to prove

(i) $T(x + y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(x)$

Proof:

$T(x) = T(a_1, a_2, a_3) = (a_2, a_3) T(y) = T(b_1, b_2, b_3) = (b_2, b_3)$.

(i) $T(x + y) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$= (a_2 + b_2, a_3 + b_3)$

$= (a_2, a_3) + (b_2, b_3)$

$= T(a_1, a_2, a_3) + T(b_1, b_2, b_3)$

$= T(x) + T(y)$

(i)$T(\alpha x) = T(\alpha a_1, \alpha a_2, \alpha a_3)$

$= (\alpha a_2, \alpha a_3)$

$= \alpha(a_2, a_3)$

$= \alpha T(a_1, a_2, a_3)$

$= \alpha T(x)$

$\therefore T$ is a linear transformation.

Example 5. Show that for $0 \leq \theta < 2\pi$,the transformation given by $T_\theta: R^2$ $R^2, T_\theta(a, b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$ is linear

Sol: $x, y \in R^2$ and $\alpha \in F$

$\therefore x = (a_1, b_1), y = (a_2, b_2) x + y = (a_1 + a_2, b_1 + b_2)$

Given

$T_\theta(a, b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$

To prove T is linear, we have to prove

(i) $T(x + y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(x)$

Proof

$T(x) = T(a_1, b_1)$

$= (a_1 \cos\theta - b_1 \sin\theta, a_1 \sin\theta + b_1 \cos\theta)$

$T(y) = T(a_2, b_2)$

$= (a_2 \cos\theta - b_2 \sin\theta, a_2 \sin\theta + b_2 \cos\theta)$

(i) $T(x + y) = T(a_1 + a_2, b_1 + b_2)$

$= \big((a_1 + a_2)\cos\theta - (b_1 + b_2)\sin\theta, (a_1 + a_2)\sin\theta + (b_1 + b_2)\cos\theta\big)$

$= (a_1 \cos\theta + a_2 \cos\theta - b_1 \sin\theta - b_2 \sin\theta, a_1 \sin\theta + a_2 \sin\theta - b_1 \cos\theta - b_2 \cos\theta)$

$= (a_1 \cos\theta - b_1 \sin\theta, a_1 \sin\theta + b_1 \cos\theta)$

$+ (a_2 \cos\theta - b_2 \sin\theta, a_2 \sin\theta - b_2 \cos\theta) = T(x) + T(y)$

(ii) $T(\alpha x) = T(\alpha a_1, \alpha b_1)$

$= (\alpha a_1 \cos\theta - \alpha b_1 \sin\theta, \alpha a_1 \sin\theta + \alpha b_1 \cos\theta)$

Example 9. Let $M(R)$ be the vector space of all 2×2 matrices over R and B be a fixed non-zero element of $M(R)$. Show that the mapping $T: M(R) \to M(R)$ defined by $T(A) = AB + BA, \forall A \in M(R)$ is a linear transformation.

Sol:

Let $A, C \in M(R)$ and $\alpha \in R$

Given

$T(A) = AB + BA, \forall A \in M(R)$ for a fixed non-zero element B of $M(R)$

To prove F is linear, we have to prove

(i)$F(A + C) = F(A) + F(C)$

(ii) $F(\alpha A) = \alpha F(A)$

Proof:

$T(A) = AB + BAT(C) = CB + BC$

(i)T(A+C)=(A+C)B+B(A+C)

$= AB + CB + BA + BC$

$= (AB + BA) + (CB + BC)$

$= T(A) + T(C)$

(ii) T(αA)=(αA)B+B(αA)

=α(AB+BA)

=αT(A)

$\therefore T$ is a linear transformation.

Example 14. Prove that there exists linear transformation $T: R^2 \to R^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

Sol: Let us express (1,1) and (2,3) as a linear combination of the standar basis vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ of R^2

$(1,1) = 1(1,0) + 1(0,1) = 1e_1 + 1e_2$

$= e_1 + e_2$ ... (1)

$(2,3) = 2(1,0) + 3(0,1) = 2e_1 + 3e_2$

$= 2e_1 + 3e_2$ ... (2)

Given

$T(1,1) = (1,0,2)$

$\Rightarrow T(e_1 + e_2) = (1,0,2)$ [from (1)]

$\Rightarrow T(e_1) + T(e_2) = (1,0,2)$ ... (3)

Also given

$T(2,3) = (1,-1,4)$

$\Rightarrow T(2e_1 + 3e_2) = (1, -1, 4)$ [from (2)]

$\Rightarrow 2T(e_1) + 3T(e_2) = (1, -1, 4)$ ...

Solve (3) and (4)

$$(3) \times 2 \Rightarrow 2T(e_1) + 2T(e_2) = (2,0,4)$$

$$(4) \quad \Rightarrow 2T(e_1) + 3T(e_2) = (1, -1, 4)$$

Subtracting $\quad -T(e_2) = (1,1,0)$

$T(e_2) = (-1, -1, 0)$

$(3) \Rightarrow T(e_1) + (-1, -1, 0) = (1, -1, 4)$

$$\Rightarrow T(e_1) = (1, -1, 4) - (-1, -1, 0)$$

$$T(e_1) = (2,0,4)$$

To find the linear transformation:

Let $(a_1, a_2) \in R^2$. Then

$$(a_1, a_2) = a_1(1,0) + a_2(0,1)$$

$$= a_1 e_1 + a_2 e_2$$

$$T(x, y) = T(a_1 e_1 + a_2 e_2)$$

$$= a_1 T(e_1) + a_2 T(e_2)$$

$$= a_1(2,0,4) + a_2(-1, -1, 0)$$

$$T(a_1, a_2) = (2a_1 - a_2, -a_2, 4a_1)$$

$T(8,11) = (5, -11, 32)$

Example 15. Is there a linear transformation $T: R^3 \to R^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

Sol: Let us express $(1,0,3)$ and $(-2,0,-6)$ as a linear combination of the R^3

standard basis vectors $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ and $e_3 = (0,1,0)$ of R^3.

$(1,0,3)= 1(1,0,0) + 0(0,1,0) + 3(0,0,1)$

$=e_1 + 0e_2 + 3e_3$

$=e_1 + 3e_3 \ldots$ (1)

$(-2,0,-6) = -2(1,0,0) + 0(0,1,0) - 6(0,0,1)$

$= -2e_1 + 0e_2 - 6e_3$

$=-2e_1 - 6e_3 \ldots$ (2)

T(1,0,3) =(1,1)

$=>T(e_1 + 3e_3) = (1,1)$ [from (1)]

$=>\mathrm{T}(e_1) + 3T(e_3) = (1,1) \ldots (3)$

Also given

$T(-2,0,-6) = (2,1)$

$T(-2e_1 - 6e_3) = (2,1))$ [from (2)]

$\Rightarrow -2T(e_1) - 6T(e_2) = (2,1)) \cdots (4)$

solve (3) and (4)

$(3) \times 2 \Rightarrow \quad 2T(e_1) + 6T(e_2) = (2,2)$

$(4) \Rightarrow -2T(e_1) - 6T(e_2) = (2,1)$

Adding $\quad 0 = (4,3)$

It is not possible.

Therefore there is no linear transtormationtion with the given data's.

Example 16. Find a linear transtormationtion $T: R^3 \rightarrow R^3$ such that T(1,1,1) = (1,1,1),T(1,2,3)=(-1,-2,3) and find T(1,1,2)=(2,2

Sol: Let us express (1,1,1), (1,2,3) and (1,1,2) as a linear combination basis vectors e_1=(1,0,0), e_2=(0,1,0) and e_3=(0,0,1) of R^3

$(1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1) = 1e_1 + 1e_2 + 1e_3$

$= e_1 + e_2 + e_3 \ldots$ (1)

$(1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1) = 1e_1 + 2e_2 + 3e_3$

$= e_1 + 2e_2 + 3e_3 \ldots$ (2)

$(1,1,2) = 1(1,0,0) + 1(0,1,0) + 2(0,0,1) = 1e_1 + 1e_2 + 2e_3$

$= e_1 + e_2 + e_3 \ldots$ (3)

Given

$T(1,1,1) = (1,1,1)$

$\Rightarrow T(e_1 + e_2 + e_3) = (1,1,1)$ [from (1)]

$\Rightarrow T(e_1) + T(e_2) + T(e_3) = (1,1,1) \ldots (4)$

Also given

T(1,2,3) =(-1,-2,3)

$\Rightarrow T(e_1 + 2e_2 + 3e_3) = (-1,-2,3)$ [from (1)

$\Rightarrow T(e_1) + 2T(e_2) + 3T(e_3) = (-1,-2,3) \ldots (5)$

Also given

$T(1,1,2) = (2,2,4)$

$\Rightarrow T(e_1 + e_2 + 2e_3) = (2,2,4)$ [from (1)]

$\Rightarrow T(e_1) + T(e_2) + 2T(e_3) = (2,2,4) \ldots (6)$

Solve (4), (5) and (6)

$$\begin{aligned}(4)\ &\Rightarrow T(e_1) + T(e_2) + T(e_3) = (1,1,1)\\ (6)\ &\Rightarrow \frac{T(e_1) + T(e_2) + 2T(e_3) = (2,2,4)}{-T(e_3) = (-1,-1,-3)}\end{aligned}$$

Subtracting

$$T(e_3) = (1,1,3)$$

(4) $\Rightarrow T(e_1) + T(e_2) + (1,1,3) = (1,1,1)$

$$\Rightarrow T(e_1) + T(e_2) = (1,1,1) - (1,1,3)$$

$$= (0,0,-2) \ldots (7)$$

(5)$\Rightarrow T(e_1) + 2T(e_2) + 3(1,1,3) = (-1,-2,3)$

$\Rightarrow T(e_1) + 2T(e_2) + (3,3,9) = (-1,-2,3)$

$$\Rightarrow T(e_1) + 2T(e_2) = (-1,-2,3) - (3,3,9)$$

$$= (-4,-5,-6) \ldots (8)$$

Solve (7) and (8)

$$(7) \Rightarrow T(e_1) + T(e_2) = (0,0,-2)$$

Subtracting(6) $\Rightarrow \frac{T(e_1)+2T(e_2)=(-4,-5,-6)}{-T(e_2)=(4,5,4)}$

$T(e_2) = (-4,-5,-4)$

(7) $\Rightarrow T(e_1) + (-4,-5,-4) = (0,0,-2)$

$$\Rightarrow T(e_1) = (0,0,-2) - (-4,-5,-4)$$

$$T(e_1) = (4,5,2)$$

To find the linear transformation:

Let $(x, y, z) \in R^3$. Then

$(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$$= xe_1 + ye_2 + ze_3$$

$r(x, y, z) \ = T(xe_1 + ye_2 + ze_3)$

$= xT(e_1) + yT(e_2) + zT(e_3)$

$= x(4,5,2) + y(-4,-5,-4) + z(1,1,3)$

$= (4x, 5x, 2x) + (-4y, -5y, -4y) + (z, z, 3z)$

$T(x, y) \ = (4x - 4y + z, 5x - 5y + z, 2x - 4y + 3z)$