3.4 Lagrange Method

3.4.1 Introduction

Another common approach for deriving the equations of motion of a system is that of using the so-called *Lagrange Method*. It originates from the sub-field of physics known as *Analytical Mechanics* [1][4], and is closely tied to both the d'Alembert and Hamilton principles, as it is one of the analytical methods used to describe the motion of physical systems. The method is centered around three fundamental concepts:

1. The definition of generalized coordinates $\mathbf{q}$ and generalized velocities $\dot{\mathbf{q}}$, which may or may not encode the information regarding the constraints applicable to the system.

2. A scalar function called the *Lagrangian* function $\mathcal{L}$. For mechanical systems, it is exactly the difference between the total kinetic energy $\mathcal{T}$ and the total potential energy $\mathcal{U}$, of the system at each instant:

$$\mathcal{L} = \mathcal{T} - \mathcal{U} \tag{3.26}$$

3. The so-called *Euler-Lagrange* equation, also known as the *Euler-Lagrange of the second kind*, which applies to the Lagrangian function $\mathcal{L}$ and to the total external generalized forces $\boldsymbol{\tau}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\right) = \boldsymbol{\tau} \tag{3.27}$$

In the most general case, the Lagrangian is a function of the generalized coordinates and velocities $\mathbf{q}$ and $\dot{\mathbf{q}}$, and it may also have an explicit dependence on time t, hence we redefine the aforementioned scalar energy functions as $\mathcal{T} = \mathcal{T}(t, \mathbf{q}, \dot{\mathbf{q}})$ and $\mathcal{U} = \mathcal{U}(t, \mathbf{q})$, thus $\mathcal{L} = \mathcal{L}(t, \mathbf{q}, \dot{\mathbf{q}})$.

In the end, one of the most notable properties of this formulation is the capacity to eliminate all internal reaction forces of the system from the final EoM, in contrast to the Newton-Euler formulation where there they are explicitly accounted for. To apply this method to derive EoM of a complex multi-body system there are additional aspects which must be considered before one can applying the three aforementioned concepts. These are presented in a concise overview at the end of this section, and are explained in the immediate continuation.

3.4.2 Kinetic Energy

The kinetic energy of a system of n_b bodies is defined as:

$$\mathcal{T} = \sum_{i=1}^{n_b} \left(\frac{1}{2} m_{i\,\mathcal{A}} \dot{\mathbf{r}}_{S_i\,\mathcal{A}}^T \dot{\mathbf{r}}_{S_i} + \frac{1}{2}\,{}_{\mathcal{B}}\mathbf{\Omega}_{S_i}^T \cdot {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\mathbf{\Omega}_{S_i} \right) \tag{3.28}$$

For every body $\mathcal{B}_i$ in the system, although the linear part may be computed while expressed in some frame $\mathcal{A}$, it may be more convenient to compute the rotational kinetic energy using expressions in another frame $\mathcal{B}$, rotated w.r.t to $\mathcal{A}$, where the inertia matrix $\mathbf{\Theta}_{S,i}$ may have a diagonal form, i.e. the basis vectors of $\mathcal{B}$ are principle w.r.t. the mass distribution. This computation will yield correct results as long as both linear and angular velocities ${}_{\mathcal{A}}\dot{\mathbf{r}}_{S,i}$ and ${}_{\mathcal{B}}\mathbf{\Omega}_{S,i}$ express the *absolute* velocities of the body, i.e velocities w.r.t. to the inertial frame.

We now need to express the kinetic energy as a function of the generalized quantities. To achieve this, we make use of the Jacobian matrices described by (2.163) and (2.164), but computed for each body $\mathcal{B}_i$ instead of the end-effector. This then allows us to use the following kinematic relationships:

$$\dot{\mathbf{r}}_{S_i} = \mathbf{J}_{S_i}\dot{\mathbf{q}} \tag{3.29}$$

$$\mathbf{\Omega}_{S_i} = \mathbf{J}_{R_i}\dot{\mathbf{q}} \tag{3.30}$$

Replacing these relationships into the definition of the kinetic energy in (3.28), results in the kinetic energy expressed in the generalized coordinates:

$$\mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\left(\sum_{i=1}^{n_b} \left(\mathbf{J}_{S_i}^T m \mathbf{J}_{S_i} + \mathbf{J}_{R_i}^T \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \right) \right)}_{\mathbf{M}(\mathbf{q})} \dot{\mathbf{q}} \tag{3.31}$$

The underlined quantity $\mathbf{M}(\mathbf{q})$ is defined as the *generalized mass matrix* or *generalized inertia matrix*, and as we will see in the continuation, is solely responsible for generating both the inertial and non-linear centrifugal and Coriolis force terms in the final EoM.