UNIT – III

APPLICATIONS OF PARTIAL DIFFERENTIALEQUATIONS

INTRODUCTION

In Science and Engineering problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with the boundary conditions constitutes a boundary value problem. In the case of ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same method is not applicable to partial differential equations because the general solution contains arbitrary constants or arbitrary functions. Hence it is difficult to adjust these constants and functions so as to satisfy the given boundary conditions. Fortunately, most of the boundary value problems involving linear partial differential equations can be solved by a simple method known as the **method of separation of variables** which furnishes particular solutions of the given differential equation directly and then these solutions can be suitably combined to give the solution of the physical problems.

Solution of the wave equation

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{-------- (1)}.$$

Let y = X(x) . T(t) be the solution of (1), where „X‟ is a function of „x‟ only and „T‟ is afunction of „t‟ only.

Then $\frac{\partial^2 y}{\partial t^2} = X T''$ and $\frac{\partial^2 y}{\partial x^2} = X'' T$.

Substituting these in (1), we get

$$X T'' = a^2 X'' T.$$

i.e, $$\frac{X''}{X} = \frac{T''}{a^2 T} \quad \text{-------- (2)}.$$

Now the left side of (2) is a function of „x‟ only and the right side is a function of „t‟ only.Since „x‟ and „t‟ are independent variables, (2) can hold good only if each side is equal toa constant.

Therefore, $$\frac{X''}{X} = \frac{T''}{a^2T} = k \text{ (say)}.$$

Hence, we get $X'' - kX = 0$ and $T'' - a^2kT = 0$ ---------------------------(3).

Solving equations (3), we get

(i) when „k‟ is positive and $k = \lambda^2$, say

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$
$$T = c_3 e^{a\lambda t} + c_4 e^{-a\lambda t}$$

(ii) when „k‟ is negative and $k = -\lambda^2$, say

$$X = c_5 \cos\lambda x + c_6 \sin\lambda x \quad T = c_7 \cos a\lambda t + c_8 \sin a\lambda t$$

(iii) when „k‟ is zero.

$$X = c_9 x + c_{10} \quad T = c_{11} t + c_{12}$$

Thus the various possible solutions of the wave equation are

$$y = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{a\lambda t} + c_4 e^{-a\lambda t}) \text{------------------------------------}(4)$$
$$y = (c_5 \cos\lambda x + c_6 \sin\lambda x)(c_7 \cos a\lambda t + c_8 \sin a\lambda t) \text{-------------------------}(5)$$
$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \text{--}(6)$$

Of these three solutions, we have to select that particular solution which suits the physical nature of the problem and the given boundary conditions. Since we are dealing with problems on vibrations of strings, „y‟ must be a periodic function of „x‟ and „t‟.

Hence the solution must involve trigonometric terms.

Therefore, the solution given by (5),

i.e, $y = (c_5 \cos\lambda x + c_6 \sin\lambda x)(c_7 \cos a\lambda t + c_8 \sin a\lambda t)$

is the only suitable solution of the wave equation.

llustrative Examples.

Example 1

If a string of length ℓ is initially at rest in equilibrium position and each of its points isgiven the velocity $\left(\frac{\partial y}{\partial t}\right)_0 = v_o \sin\frac{\pi x}{\ell}$, $0 < x < \ell$. Determine the displacement y(x,t).t =

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{------------------------} (1)$$

The boundary conditions are

i. $y(0,t) = 0$, for $t \geq 0$.
ii. $y(\ell,t) = 0$, for $t \geq 0$.
iii. $y(x,0) = 0$, for $0 \leq x \leq \ell$.

iv. $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_o \sin\frac{\pi x}{\ell}$, for $0 \leq x \leq \ell$.t

Since the vibration of a string is periodic, therefore, the solution of (1) is of the formy(x,t)

$$= (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{------------------------------} (2)$$

Using (i) in (2) , we get

$$0 = A(C\cos\lambda at + D\sin\lambda at)\text{ , for all } t \geq 0.$$

Therefore, $A = 0$

Hence equation (2) becomes

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \text{ ---------------------(3)}$$

Using (ii) in (3), we get

$0 = B\sin\lambda\ell\,(C\cos\lambda at + D\sin\lambda at)$, for all $t \geq 0$, which gives $\lambda\ell = n\pi.n\pi$

Hence, $\lambda = \dfrac{\quad}{\quad}$, n being an integer.ℓ

$$\text{Thus, } y(x,t) = B\sin\frac{n\pi x}{\ell}\left(C\cos\frac{n\pi at}{\ell} + D\sin\frac{n\pi at}{\ell}\right) \text{ ------------------------(4)}$$

Using (iii) in (4), we get

$$0 = B\sin\frac{n\pi x}{\ell}.\,C$$

which implies $C = 0$.

$$\therefore \quad y(x,t) = B\sin\frac{n\pi x}{\ell}.\quad D\sin\frac{n\pi at}{\ell}$$

$$= B_1\sin\frac{n\pi x}{\ell}.\sin\frac{n\pi at}{\ell}, \text{ where } B_1 = BD.\ell$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}\sin\frac{n\pi at}{\ell} \text{ --------------------------------(5)}$$

Differentiating (5) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}.\cos\frac{n\pi at}{\ell} \cdots\cdots\cdots \frac{n\pi a}{\ell}$$

Using condition (iv) in the above equation, we get

$$v_o \sin\frac{\pi x}{\ell} = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi a}{\ell} \cdot \sin\frac{n\pi x}{\ell}$$

i.e, $$v_o \sin\frac{\pi x}{\ell} = B_1 \cdot \frac{\pi a}{\ell} \cdot \sin\frac{\pi x}{\ell} + B_2 \cdot \frac{2\pi a}{\ell} \cdot \sin\frac{2\pi x}{\ell} + \cdots\cdots$$

Equating like coefficients on both sides, we get

$$B_1 \frac{\pi a}{\ell} = v_o, \quad B_2 \cdot \frac{2\pi a}{\ell} = 0, \quad B_3 \frac{3\pi a}{\ell} = 0, \cdots\cdots\cdots$$

i.e, $$B_1 = \frac{v_o\ell}{\pi a}, \quad B_2 = B_3 = B_4 = B_5 = \cdots\cdots = 0.$$

Substituting these values in (5), we get the required solution. $v_o\ell$

i.e, $$y(x,t) = \frac{\quad}{\pi a} \sin\frac{\pi x}{\ell} \cdot \sin\frac{\pi at}{\ell}$$

Example 2

A tightly stretched string with fixed end points x = 0 & x = ℓ is initially at rest in its equilibrium position . If it is set vibrating by giving to each of its points avelocity $\partial y/\partial t = kx(\ell-x)$ at t = 0. Find the displacement y(x,t).

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{-----------------------}(1)$$

The boundary conditions are

i. $y(0,t) = 0$, for $t \geq 0$.

ii. $y(\ell,t) = 0$, for $t \geq 0$.

iii. $y(x,0) = 0$, for $0 \leq x \leq \ell$.

iv. $\left(\dfrac{\partial y}{\partial t}\right)_0 = kx(\ell - x)$, for $0 \leq x \leq \ell$.t =

Since the vibration of a string is periodic, therefore, the solution of (1) is of the formy(x,t) =

$$(A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ------------------------------ } (2)$$

Using (i) in (2), we get

$$0 = A(C\cos\lambda at + D\sin\lambda at), \text{ for all } t \geq 0.$$

which gives $A = 0$.

Hence equation (2) becomes

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \text{ -------------------- } (3)$$

Using (ii) in (3), we get

$$0 = B\sin\lambda\ell(C\cos\lambda at + D\sin\lambda at), \text{ for all } t \geq 0.\text{which}$$

implies $\lambda\ell = n\pi$.

Hence, $\lambda = \dfrac{n\pi}{\ell}$, n being an integer.

Thus, $y(x,t) = B\sin\dfrac{n\pi x}{\ell}\left(C\cos\dfrac{n\pi at}{\ell} + D\sin\dfrac{n\pi at}{\ell}\right)$ ------------------------------ (4)

Using (iii) in (4), we get

$$0 = B\sin\frac{n\pi x}{\ell}.C$$

Therefore, $C = 0$.

Hence, $y(x,t) = B\sin\dfrac{n\pi x}{\ell}.\ D\sin\dfrac{n\pi at}{\ell}$

$$= B_1\sin\frac{n\pi x}{\ell}.\sin\frac{n\pi at}{\ell}, \text{ where } B_1 = BD.\ell$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}\sin\frac{n\pi at}{\ell} \quad \text{------------------------------ (5)}$$

Differentiating (5) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} .\cos\frac{n\pi at}{\ell} . \frac{n\pi a}{\ell}$$

Using (iv), we get

$$kx(\ell-x) = \sum_{n=0}^{\infty} B_n.\frac{n\pi a}{\ell} . \sin\frac{n\pi x}{\ell}$$

$$\text{i.e,}\quad B_n.\frac{n\pi a}{\ell} = \frac{2}{\ell}\int_0^{\ell} f(x). \sin\frac{n\pi x}{\ell}\, dx$$

$$\text{i.e,}\quad B_n = \frac{2}{n\pi a}\int_0^{\ell} f(x). \sin\frac{n\pi x}{\ell}\, dx$$

$$= \frac{2}{n\pi a}\int_0^{\ell} kx(\ell-x)\ \sin\frac{n\pi x}{\ell}\, dx$$

$$= \frac{2k}{n\pi a}\int_0^{\ell} (\ell x - x^2)\, d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi x}{\ell}}\right)$$

$$= \frac{2k}{n\pi a}\quad (\ell x- x^2)\, d\left(\frac{-\cos\ \text{—}}{\frac{n\pi}{\ell}}\right) - (\ell-2x)\left(\ \text{—}\ \right.$$

$$= \frac{2k}{n\pi a}\left\{\frac{-2\cos n\pi}{\frac{n^3\pi^3}{\ell^3}} + \frac{2}{\frac{n^3\pi^3}{\ell^3}}\right\}$$

$$= \frac{2k}{n\pi a} \cdot \frac{2\ell^3}{n^3\pi^3}\{1 - \cos n\pi\}$$

i.e, $$B_n = \frac{4k\ell^3}{n^4\pi^4 a}\{1 - (-1)^n\}$$

or $$B_n = \begin{cases} \dfrac{8k\ell^3}{n^4\pi^4 a}, & \text{if n is odd} \\ 0. & \text{if n is even} \end{cases}$$

Substituting in (4), we get

$$y(x,t) = \sum_{n=1,3,5,\ldots\ldots}^{\infty} \frac{8k\ell^3}{n^4\pi^4 a} \sin\frac{n\pi at}{\ell} \sin\frac{n\pi x}{\ell}$$

Therefore the solution is

$$y(x,t) = \frac{8k\ell^3}{\pi^4 a} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi at}{\ell} \sin\frac{(2n-1)\pi x}{\ell}$$

Example 3

A tightly stretched string with fixed end points $x = 0$ & $x = \ell$ is initially in a position given by $y(x,0) = y_0\sin^3(\pi x/\ell)$. If it is released from rest from this position, findthe displacement y at any time and at any distance from the end $x = 0$.

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{----------------------} (1)$$

The boundary conditions are(i)
$y(0,t) = 0, \forall\ t \geq 0.$
(ii) $y(\ell,t) = 0, \forall\ t \geq 0.$(iii)

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \text{ for } 0 < x < \ell.$$

(iv) $y(x,0) = y_0 \sin^3((\pi x/\ell)$, for $0 < x < \ell$.

The suitable solution of (1) is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ----------------------- } (2)$$

Using (i) and (ii) in (2) , we get

$$A = 0 \ \& \ \lambda = \frac{n\pi}{\ell}$$

$$\therefore \quad y(x,t) = B\sin\frac{n\pi x}{\ell}\left(C\cos\frac{n\pi at}{\ell} + D\sin\frac{n\pi at}{\ell}\right) \text{ -------------- } (3)$$

$$\text{Now, } \frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{\ell}\left[-C\sin\frac{n\pi at}{\ell}\cdot\frac{n\pi a}{\ell} + D\cos\frac{n\pi at}{\ell}\cdot\frac{n\pi a}{\ell}\right]$$

Using (iii) in the above equation , we get

$$0 = B\sin\frac{n\pi x}{\ell} \ D\frac{n\pi a}{\ell}$$

Here, B can not be zero . Therefore D = 0.

Hence equation (3) becomes

$$y(x,t) = B\sin\frac{n\pi x}{\ell}\cdot C\cos\frac{n\pi at}{\ell}$$

$$= B_1\sin\frac{n\pi x}{\ell}\cdot\cos\frac{n\pi at}{\ell}, \text{ where } B_1 = BC\ell$$

The most general solution is

∞ $\quad n\pi x \quad n\pi at$

$$y(x,t) = \sum_{n=1} B_n \sin \frac{\quad}{\ell} \cos \frac{\quad}{\ell} \text{ ------------------------(4)}$$

Using (iv), we get

$$y_0 \sin^3 \frac{n\pi}{\ell} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\ell}$$

$$\text{i.e, } \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{\ell} = y_0 \left\{ \frac{3}{4} \sin \frac{\pi x}{\ell} - \frac{1}{4} \sin \frac{3\pi x}{\ell} \right\}$$

$$\text{i.e, } B_1 \sin \frac{\pi x}{\ell} + B_2 \sin \frac{2\pi x}{\ell} + B_3 \sin \frac{3\pi x}{\ell} + \ldots$$

$$= \frac{3y_0}{4} \sin \frac{\pi x}{\ell} - \frac{y_0}{4} \sin \frac{3\pi x}{\ell}$$

Equating the like coefficients on both sides, we get

$$B_1 = \frac{3y_0}{4}, \; B_3 = \frac{-y_0}{4}, \; B_2 = B_4 = \ldots = 0 .$$

Substituting in (4), we get

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{\ell} \cdot \cos \frac{\pi at}{\ell} - \frac{y_0}{4} \sin \frac{3\pi x}{\ell} \cdot \cos \frac{3\pi at}{\ell}$$

Example 4

A string is stretched & fastened to two points $x = 0$ and $x = \ell$ apart. Motion is started by displacing the string into the form $y(x,0) = k(\ell x - x^2)$ from which it is released at time $t = 0$. Find the displacement $y(x,t)$.

Solution

The displacement $y(x,t)$ is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{ ------------------------(1)}$$

The boundary conditions are

(i) $y(0,t) = 0, \quad \forall\, t \geq 0.$

(ii) $y(\ell,t) = 0, \quad \forall\, t \geq 0.$

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_0 = 0$, for $0 < x < \ell.t =$

(iv) $y(x,0) = k(\ell x - x^2)$, for $0 < x < \ell$.

The suitable solution of (1) is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ----------------------- } (2)$$

Using (i) and (ii) in (2) , we get

$$A = 0 \quad \& \quad \lambda = \frac{n\pi}{\ell}.$$

$$\therefore \quad y(x,t) = B\sin\frac{n\pi x}{\ell}\left(C\cos\frac{n\pi at}{\ell} + D\sin\frac{n\pi at}{\ell}\right) \text{--------------} (3)$$

$$\text{Now, } \frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{\ell}\left(-C\sin\frac{n\pi at}{\ell}\cdot\frac{n\pi a}{\ell} + D\cos\frac{n\pi at}{\ell}\cdot\frac{n\pi a}{\ell}\right)$$

Using (iii) in the above equation , we get

$$0 = B\sin\frac{n\pi x}{\ell}\, D\,\frac{n\pi a}{\ell}$$

Here, B can not be zero

$D = 0$

Hence equation (3) becomes

$$y(x,t) = B\sin\frac{n\pi x}{\ell}\,.\,C\cos\frac{n\pi at}{\ell}$$

$$= B_1\sin\frac{n\pi x}{\ell}\,.\,\cos\frac{n\pi at}{\ell}, \text{ where } B_1 = BC\ell$$

The most general solution is

$$y(x,t) = \sum^{\infty} B_n\sin\frac{n\pi x}{} \cos\frac{n\pi at}{} \text{ ------------------------------- } (4)$$

$n=1 \qquad \ell \qquad \ell$

Using (iv), we get $$kx(\ell x - x^2) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \text{ ------------------------------(5)}$$

The RHS of (5) is the half range Fourier sine series of the LHS function .

$$\therefore B_n = \frac{2}{\ell}\int_0^{\ell} f(x) \,.\, \sin\frac{n\pi x}{\ell}\, dx$$

$$= \frac{2k}{\ell}\int_0^{\ell} (\ell x - x^2)\, d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n}{\pi}}\right)$$

$$= \frac{2k}{\ell}\Bigg\{(\ell x - x^2)\, d\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right) - (\ell - 2x)\left(\frac{-\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right)$$

$$+ (-2)\left(\frac{\cos\frac{n\pi x}{\ell}}{\frac{n^3\pi^3}{\ell^3}}\right)\Bigg\}_0^{\ell}$$

$$= \frac{2k}{\ell}\left\{\frac{-2\cos n\pi}{\frac{n^3\pi^3}{\ell^3}} + \frac{2}{\frac{n^3\pi^3}{\ell^3}}\right\}$$

$$= \frac{2k}{\ell} \cdot \frac{2\ell^3}{n^3\pi^3}\{1 - \cos n\pi\}\ell$$

i.e, $$B_n = \frac{4k\ell^2}{n^3\pi^3}\{1 - (-1)^n\}$$

$$\text{or} \quad B_n = \begin{cases} \dfrac{8k\ell^2}{n^3\pi^3}, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases}$$

$$\therefore y(x,t) = \sum_{n=odd}^{\infty} \frac{8k\ell^2}{n^3\pi^3} \cos\frac{n\pi at}{\ell} . \sin\frac{n\pi x}{\ell}$$

$$\text{or} \quad y(x,t) = \frac{8k}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos\frac{(2n-1)\pi at}{\ell} . \sin\frac{(2n-1)\pi x}{\ell}$$

Example 5

A uniform elastic string of length 2ℓ is fastened at both ends. The midpoint of the string is taken to the height „b" and then released from rest in that position . Find the displacement of the string.

Solution

The displacement y(x,t) is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \text{------------------------} (1)$$

The suitable solution of (1) is given by

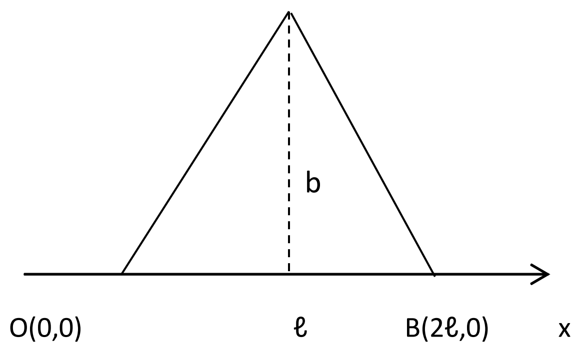
$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{------------------------} (2)$$

The boundary conditions are(i)

$y(0,t) = 0, \forall\ t \geq 0.$

(ii) $y(\ell,t) = 0, \quad \forall\ t \geq 0.$

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_0 = 0$, for $0 < x < 2\ell.t =$



$$\text{(iv) } y(x,0) = \begin{cases} (b/\ell)x, & 0<x<\ell \\ -(b/\ell)(x-2\ell), & \ell<x<2\ell \end{cases}$$

[Since, equation of OA is $y = (b/\ell)x$ and equation of AB is $(y-b)/(o-b) = (x-\ell)/(2\ell-\ell)$]

Using conditions (i) and (ii) in (2), we get

$$A = 0 \quad \& \quad \lambda = \frac{n\pi}{2\ell}$$

$$\therefore \quad y(x,t) = B \sin\frac{n\pi x}{2\ell}\left(C\cos\frac{n\pi at}{2\ell} + D\sin\frac{n\pi at}{2\ell}\right) \text{------------- (3)}$$

$$\text{Now, } \frac{\partial y}{\partial t} = B \sin\frac{n\pi x}{2\ell}\left(-C\sin\frac{n\pi at}{2\ell}\cdot\frac{n\pi a}{2\ell} + D\cos\frac{n\pi at}{2\ell}\cdot\frac{n\pi a}{2\ell}\right)$$

Using (iii) in the above equation , we get

$$0 = B \sin\frac{n\pi x}{2\ell}\, D\,\frac{n\pi a}{2\ell}$$

Here B can not be zero, therefore D = 0.
Hence equation (3) becomes

$$y(x,t) = B \sin\frac{n\pi x}{2\ell} \cdot C\cos\frac{n\pi at}{2\ell}$$

$n\pi x$ $\quad$ $n\pi at$

$$= B_1 \sin \frac{\quad}{\quad} \cdot \cos \frac{\quad}{2\ell}, \text{ where } B_1 = BC2\ell$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2\ell} \cos \frac{n\pi at}{2\ell} \quad \text{------------------ (4)}$$

Using (iv), We get

$$y(x,0) = \sum_{n=1}^{\infty} B_n . \sin \frac{n\pi x}{2\ell} \quad \text{------------------ (5)}$$

The RHS of equation (5) is the half range Fourier sine series of the LHS function .

$$\therefore B_n = \frac{2}{2\ell} \int_0^{2\ell} f(x) . \sin \frac{n\pi x}{2\ell} dx$$

$$= \frac{1}{\ell} \left[\int_0^{\ell} f(x) . \sin \frac{n\pi x}{2\ell} dx + \int_{\ell}^{2\ell} f(x) . \sin \frac{n\pi x}{2\ell} dx \right]$$

$$= \frac{1}{\ell} \left[\int_0^{\ell} \frac{b}{\ell} x \sin \frac{n\pi x}{2\ell} dx + \int_{\ell}^{2\ell} \frac{-b}{\ell} (x-2\ell) \sin \frac{n\pi x}{2\ell} dx \right]$$

$$= \frac{1}{\ell} \left\{ \frac{b}{\ell} \int_0^{\ell} x d \left(\frac{-\cos \frac{n\pi x}{2\ell}}{\frac{n\pi}{2\ell}} \right) - \frac{b}{\ell} \int_{\ell} (x-2\ell) d \left(\frac{-\cos \frac{n\pi x}{\ell}}{\frac{n\pi}{2\ell}} \right) \right.$$

$$= \frac{1}{\ell} \left\{ \frac{b}{\ell} \left[(x) \left(\frac{-\cos \frac{n\pi x}{2\ell}}{\frac{n\pi}{2\ell}} \right) - (1) \left(\frac{-\sin \frac{n\pi}{\quad}}{\quad} \right) \right]_0^{\ell} \right.$$

$$= \frac{b}{\ell^2}\left\{ \frac{-\ell\cos\frac{n\pi}{2}}{\frac{n\pi}{2\ell}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4\ell^2}} + \frac{\ell\cos\frac{n\pi}{2}}{\frac{n\pi}{2\ell}} + \frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4\ell^2}} \right\}$$

$$= \frac{8b\sin(n\pi/2)}{n^2\pi^2}$$

Therefore the solution is

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8b\sin(n\pi/2)}{n^2\pi^2} \cos\frac{n\pi at}{2\ell} \sin\frac{n\pi x}{2\ell}$$

Example 6

A tightly stretched string with fixed end points $x = 0$ & $x = \ell$ is initially in the position $y(x,0) = f(x)$. It is set vibrating by giving to each of its points avelocity $\frac{\partial y}{\partial t} = g(x)$ at $t = 0$. Find the displacement $y(x,t)$ in the form of Fourier series.

Solution

The displacement $y(x,t)$ is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{-----------------------(1)}$$

The boundary conditions are(i)

$y(0,t) = 0, \forall\, t \geq 0.$

(ii) $y(\ell,t) = 0, \quad \forall\, t \geq 0.$

(iii) $y(x,0) = f(x)$, for $0 \leq x \leq \ell$.

(iv) $\left(\dfrac{\partial u}{\partial t}\right)_0 = g(x)$, for $0 \leq x \leq \ell.t =$

The solution of equation .(1) is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \text{ ----------------------- } (2)$$

where A, B, C, D are constants.

Applying conditions (i) and (ii) in (2), we have

$$A = 0 \quad \text{and} \quad \lambda = \frac{n\pi}{\ell}.$$

Substituting in (2), we get

$$y(x,t) = B\sin\frac{n\pi x}{\ell}\left(C\cos\frac{n\pi at}{\ell} + D\sin\frac{n\pi at}{\ell}\right)$$

$$y(x,t) = \sin\frac{n\pi x}{\ell}\left(B_1\cos\frac{n\pi at}{\ell} + D_1\sin\frac{n\pi at}{\ell}\right) \text{ where } B_1 = BC \text{ and } D_1 = BD.\ell$$

The most general solution. is

$$y(x,t) = \sum_{n=1}^{\infty} \left. B_n\cos\frac{n\pi at}{\ell} + D_n.\sin\frac{n\pi a}{t\ell}\right] .\sin\frac{n\pi x}{\ell} \text{ ---------------------- } (3)$$

Using (iii), we get

$$f(x) = \sum_{n=1}^{\infty} B_n . \sin \frac{n\pi x}{\ell} \text{-------------------------------}(4)$$

The RHS of equation (4) is the Fourier sine series of the LHS function.

$$\therefore B_n = \frac{2}{\ell} \int_0^{\ell} f(x) . \sin \frac{n\pi x}{\ell} \, dx$$

Differentiating (3) partially w.r.t „t", we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(-B_n \sin \frac{n\pi at}{\ell} \frac{n\pi a}{\ell} + D_n . \cos \frac{n\pi at}{\ell} \frac{n\pi a}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

Using condition (iv) , we get

$$g(x) = \sum_{n=1}^{\infty} \left(D_n \frac{n\pi a}{\ell} \right) . \sin \frac{n\pi x}{\ell} \text{-----------------------}(5)$$

The RHS of equation (5) is the Fourier sine series of the LHS function.

$$\therefore D_n . \frac{n\pi a}{\ell} = \frac{2}{\ell} \int_0^{\ell} g(x) . \sin \frac{n\pi x}{\ell} \, dx$$

$$\Rightarrow D_n = \frac{2}{n\pi a} \int_0^{\ell} g(x) . \sin \frac{n\pi x}{\ell} \, dx$$

Substituting the values of B_n and D_n in (3), we get the required solution of thegiven equation.

Exercises

(1) Find the solution of the equation of a vibrating string of length „ℓ", satisfying theconditions

$y(0,t) = y(\ell,t) = 0$ and $y = f(x)$, $\partial y/ \partial t = 0$ at $t = 0$.

(2) A taut string of length 20 cms. fastened at both ends is displaced from its position ofequilibrium, by imparting to each of its points an initial velocity given by

$v = x$ in $0 \le x \le 10$

$= 20 - x$ in $10 \le x \le 20$,

„x" being the distance from one end. Determine the displacement at any subsequent time.

(3) Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

corresponding to the triangular initial deflection

$f(x) = (2k/\ell)\, x$ when $0 < x < \ell/2$

$= (2k/\ell)(\ell - x)$ when $\ell/2 < x < \ell$,

and initial velocity zero.

(4) A tightly stretched string with fixed end points $x = 0$ and $x = \ell$ is initially at rest in itsequilibrium position. If it is set vibrating by giving to each of its points a velocity $\partial y/\partial t = f(x)$ at $t = 0$. Find the displacement $y(x,t)$.

(5) Solve the following boundary value problem of vibration of string

i. $y(0,t) = 0$

ii. $y(\ell,t) = 0$

iii. $\frac{\partial y}{\partial t}(x,0) = x(x - \ell)$, $0 < x < \ell$.

iv. $y(x,0) = x$ in $0 < x < \ell/2$

$= \ell - x$ in $\ell/2 < x < \ell$.

(6) A tightly stretched string with fixed end points $x = 0$ and $x = \ell$ is initially in a position given by $y(x,0) = k(\sin(\pi x/\ell) - \sin(2\pi x/\ell))$. If it is released from rest, find thedisplacement of „y" at any distance „x" from one end at any time „t".