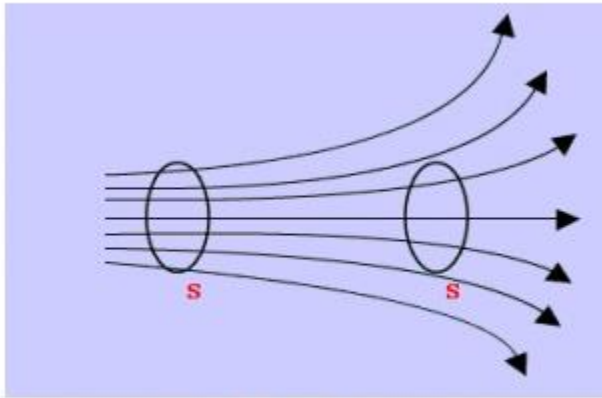# Divergence of a Vector Field

In study of vector fields, directed line segments, also called flux lines or streamlines, represent field variations graphically. The intensity of the field is proportional to the density of lines. For example, the number of flux lines passing through a unit surface S normal to the vector measures the vector field strength.

Fig 1.18: Flux Lines

We have already defined flux of a vector field as

$$\psi = \int_S A \cos \theta \, ds = \int_S \vec{A} \cdot \hat{a}_n \, ds = \int_S \vec{A} \cdot d\vec{s}$$

....(1.57)

For a volume enclosed by a surface,

$$\psi = \oint_S \vec{A} \cdot d\vec{s}$$

....(1.58)

We define the divergence of a vector field $\vec{A}$ at a point P as the net outward flux from a volume enclosing P, as the volume shrinks to zero.

$$div \ \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}$$

....(1.59)

Here ΔV is the volume that encloses P and S is the corresponding closed surface.
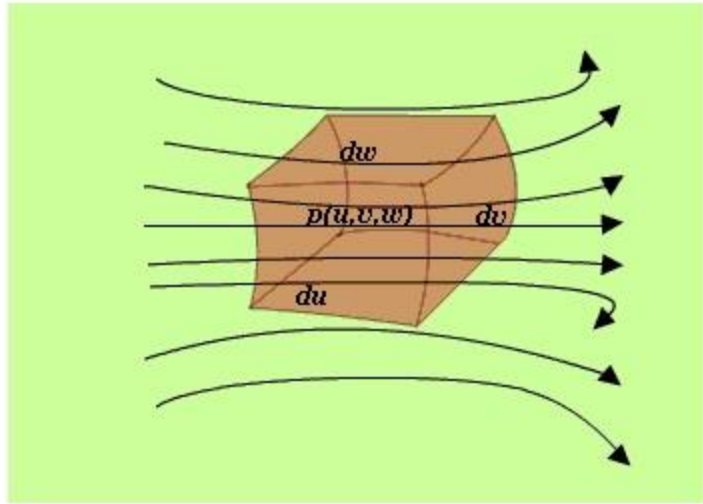
Fig 1.19: Evaluation of divergence in curvilinear coordinate

Let us consider a differential volume centered on point P(u,v,w) in a vector field $\vec{A}$. The flux through an elementary area normal to u is given by ,

$$\phi_u = \vec{A} \cdot \hat{a}_u h_2 h_3 dvdw \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1.60)$$

Net outward flux along u can be calculated considering the two elementary surfaces perpendicular to u .

$$\left[h_2 h_3 A_u \Big|_{\left(u+\frac{du}{2},v,w\right)} - h_2 h_3 A_u \Big|_{\left(u-\frac{du}{2},v,w\right)} \right] dvdw \cong \frac{\partial\left(h_2 h_3 A_u\right)}{\partial u} dudvdw$$

......................................(1.61)

Considering the contribution from all six surfaces that enclose the volume, we can write

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta v} = \frac{dudvdw \frac{\partial\left(h_2 h_3 A_u\right)}{\partial u} + dudvdw \frac{\partial\left(h_1 h_3 A_v\right)}{\partial v} + dudvdw \frac{\partial\left(h_1 h_2 A_w\right)}{\partial w}}{h_1 h_2 h_3 dudvdw}$$

$$\therefore \nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial\left(h_2 h_3 A_u\right)}{\partial u} + \frac{\partial\left(h_1 h_3 A_v\right)}{\partial v} + \frac{\partial\left(h_1 h_2 A_w\right)}{\partial w} \right]$$

...........................(1.62)

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for divergence written

In Cartesian coordinates:

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

...........................(1.63)

In cylindrical coordinates:

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

...........................(1.64)

and in spherical polar coordinates:

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

...........................(1.65)

In connection with the divergence of a vector field, the following can be noted

- Divergence of a vector field gives a scalar.

$$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$$

- $$\nabla \cdot (V\vec{A}) = V\nabla \cdot \vec{A} + \vec{A} \cdot \nabla V \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1.66)$$

Divergence theorem :

Divergence theorem states that the volume integral of the divergence of vector field is equal to the net outward flux of the vector through the closed surface that bounds the volume. Mathematically,

$$\int_V \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot d\vec{s}$$

Proof:

Let us consider a volume V enclosed by a surface S. Let us subdivide the volume in large number of cells. Let the kth cell has a volume $\Box V_k$ and the corresponding surface is denoted by Sk. Interior to the volume, cells have common surfaces. Outward flux through these common surfaces from one cell becomes the inward flux for the neighboring cells. Therefore when the total flux from these cells are considered, we actually get the net outward flux through the surface surrounding the volume. Hence we can write:

$$\oint_S \vec{A} \cdot d\vec{s} = \sum_k \oint_{S_k} \vec{A} \cdot d\vec{s} = \sum_k \frac{\oint_{S_k} \vec{A} \cdot d\vec{s}}{\Delta V_k} \Delta V_k \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(1.67)$$

In the limit, that is when $K \rightarrow \infty$ and $\Delta V_k \rightarrow 0$ the right hand of the expression can be written as

$$\int_V \nabla \cdot A\, dV$$

Hence we get $\oint_S \vec{A} \cdot d\vec{S} = \int_V \nabla \cdot A\, dV$, which is the divergence theorem.