DECIMATION AND INTERPOLATION

Systems that employ multiple sampling rates in the processing of digital signals are called Multirate Digital Signal Processing Systems.

The two primary operations used in multirate signal processing are:

- Decimation
- Interpolation

Decimation

Decimation reduces the sampling rate effectively by compressing the data and retaining the desired information.

Interpolation

Interpolation increases the sampling rate.

Sampling Rate Conversion:

The process of converting a digital signal from a given sampling rate to a different sampling rate is called Sampling Rate Conversion.

Decimator (Down-Sampler):

Let $x(n)$ be a sequence which has been sampled at rate 1 (unity) ,ie $x(n)$ is obtained by sampling a continuous time sequence $x(t)$ at Nyquist rate

$$x(n) = {x(t)}/_{t = n}$$

The decimation operator ↓D converts the input sequence $x(n)$ into a new sequence $y(n)$,having the rate $\frac{1}{D}$.

$$y(n) = x(Mn)$$

ie, $y(n)$ contains every D^{th} sample from $x(n)$.

$$y(n) = {x(t)}/_{t = Dn}$$

The downsampler is represented by the following diagram

Problem:

1. For the given data sequence $x(n) = \{1,4,6,8,10,12,13,2,3,15,5\}$ find the output sequence which is down sampled version of $x(n) by\ (i) 2\ (ii) 3$.

Soln:

$(i) y(n) = (\downarrow 2) x(n)$

The down sampler simply keeps every second sample and discards the others.

$$y(n) = \{1,6,10,13,3,5\}$$

$(ii) y(n) = (\downarrow 3) x(n)$

The down sampler simply keeps everyThird sample and discards the others.

$$y(n) = \{1,8,13,15\}$$

DECIMATION BY A FACTOR D:

Decimation by a factor D, means to reduce the sampling rate.

Let us consider that $x(n)$ is having a spectrum $X(\omega)\ with\ 0 \le |\omega| \le \pi$. Let the sampling frequency (F_x) and maximum frequency (F_{max}) be related as

$$f_x \ge 2f_{max}$$

$$f_{max} \le \frac{f_x}{2}$$

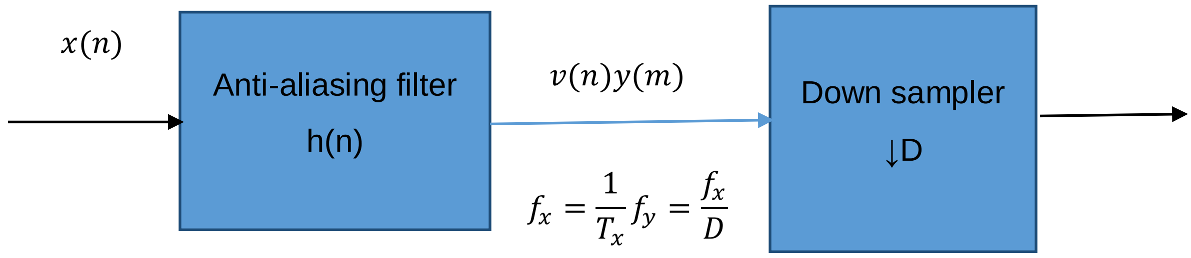


Fig:Decimation by a factor D

Derivation of Decimation Equation:

The output of the filter is a sequence $v(n)$ given by

$$v(n) = x(n) * h(n)$$

$$v(n) = \sum_{k=0}^{\infty} h(k) x(n-k) ----(1)$$

When $v(n)$ is down sampled by the factor D, we get $y(m)$.

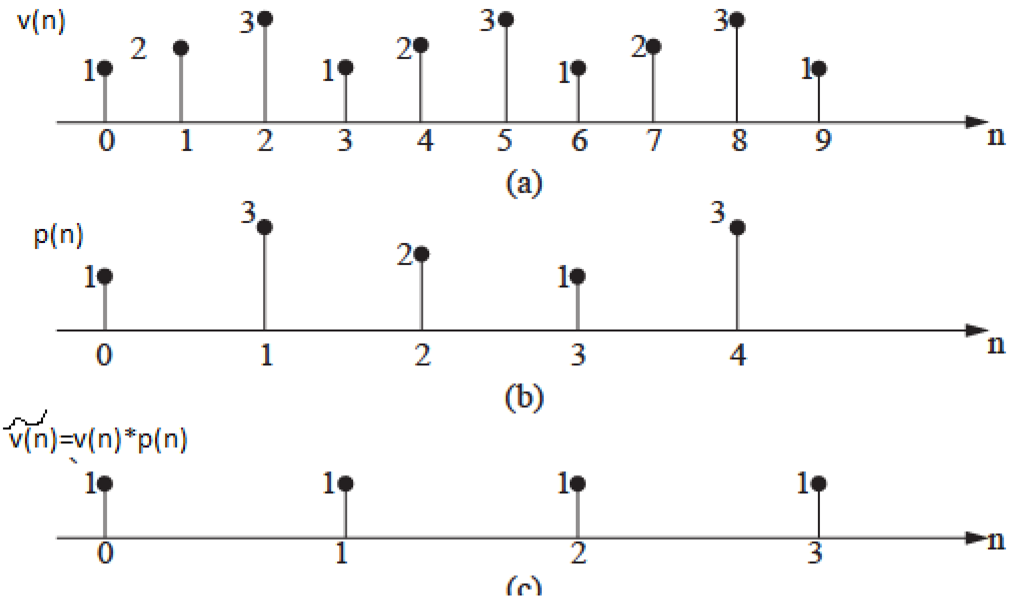
$$y(m) = v(mD)$$

$n = mD$

$$y(m) = \sum_{k=0}^{\infty} h(k)x(mD-k) ------ (2)$$

The equation gives the output signal.

Relationship between the spectrum of $x(n)$ and $y(n)$



Step (1):

The down sampled sequence is given by the product of $v(n)$ with sampling sequence $p(n)$.In the above figure it is observed that $p(n)$ picks up every 3^{rd} sample (ie,D=3)

$$\tilde{v}(n) = v(n)p(n) --- (1)$$

Where $v(n)$ is the output of the filter

$p(n)$ is the periodic train of impulses.

$$y(m) = \tilde{v}(mD) --- (2)$$

Step (2):

Taking Z transform of equ (2) we get,

$$Y(Z) = \sum_{m=-\infty}^{\infty} y(m)Z^{-m}$$

$$Y(Z) = \sum_{m=-\infty}^{\infty} \tilde{v}(mD)Z^{-m} ----(3)$$

Let $mD = p$

$m = -\infty \ , p = -\infty$

$m = \infty \ , p = \infty$

$Y(Z) = \sum_{p=-\infty}^{\infty} \tilde{v}(p) Z^{-p/D}$

Changing the dummy variable p to m we get,

$$Y(Z) = \sum_{m=-\infty}^{\infty} \tilde{v}(m) Z^{-m/D} --- (4)$$

<u>Step (3):</u>

From equ (1) we know that,

$$\tilde{v}(n) = v(n)p(n)$$

$Let\ n = m$

$$\tilde{v}(m) = v(m)p(m) --- (5)$$

Sub Equ (5) in (4) we get,

$$Y(Z) = \sum_{m=-\infty}^{\infty} v(m)p(m) Z^{-m/D} --- (6)$$

Where $p(m)$

Is the periodic train of impulses and represented by discrete time Fourier series as

$$p(m) = \frac{1}{D}\sum_{k=0}^{D-1} e^{\frac{j2\pi km}{D}} --- (7)$$

Sub equ (7) in (6) we get,

$$Y(Z) = \sum_{m=-\infty}^{\infty} v(m) \frac{1}{D}\sum_{k=0}^{D-1} e^{\frac{j2\pi km}{D}} Z^{-m/D}$$

Interchanging the order of summation

$$Y(Z) = \frac{1}{D}\sum_{k=0}^{D-1}\left[\sum_{m=-\infty}^{\infty} v(m)\left(e^{\frac{-j2\pi k}{D}} Z^{1/D}\right)^{-m}\right]$$

$$Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} v\left(e^{\frac{-j2\pi k}{D}} Z^{1/D}\right) ----(8)$$

We know that,

$$v(n) = x(n) * h(n)$$

Taking Z transform on both sides,

$$V(Z) = X(Z)H(Z)$$

$Z = e^{\frac{-j2\pi k}{D}} Z^{1/D}$

$$V\left(e^{\frac{-j2\pi k}{D}} Z^{1/D}\right) = X\left(e^{\frac{-j2\pi k}{D}} Z^{1/D}\right) H\left(e^{\frac{-j2\pi k}{D}} Z^{1/D}\right) --- (9)$$

Sub equ (9) in (8) we get,

$$Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{\frac{-j2\pi k}{D}} Z^{1/D}\right) H\left(e^{\frac{-j2\pi k}{D}} Z^{1/D}\right) ----(10)$$

Step (4):

To find the spectrum of $y(m)$,sub $Z = e^{j\omega_y}$ where ω_y represents the frequency of $y(m)$.

$$Y\left(e^{j\omega_y}\right) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{\frac{-j2\pi k}{D}} e^{\frac{j\omega_y}{D}}\right) H\left(e^{\frac{-j2\pi k}{D}} e^{\frac{j\omega_y}{D}}\right)$$

$$Y\left(e^{j\omega_y}\right) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{\frac{\left(j(\omega_y-2\pi k)\right)}{D}}\right) H\left(e^{\frac{\left(j(\omega_y-2\pi k)\right)}{D}}\right)$$

$H\left(e^{j\omega}\right) = H(\omega)$

$$Y(e^{\omega_y}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{\frac{\left((\omega_y-2\pi k)\right)}{D}}\right) H\left(e^{\frac{\left((\omega_y-2\pi k)\right)}{D}}\right) ----(11)$$

This equation indicates the replicas of $X\left(\frac{\omega_y}{D}\right)$ at 0,2π,4π,6π..

Step (5):

Let K=0,

Equ (11) can be written as,

$$Y\left(\omega_y\right) = \frac{1}{D} X\left(\frac{\omega_y}{D}\right) H\left(\frac{\omega_y}{D}\right)$$

Step (6):

With properly designed $H\left(\frac{\omega_y}{D}\right)$,the aliasing is eliminated.

If we sub $H\left(\frac{\omega_y}{D}\right) = 1$ then,

$$Y\left(\omega_y\right) = \frac{1}{D} X\left(\frac{\omega_y}{D}\right) \; for \; 0 \leq \left|\omega_y\right| \leq \pi$$

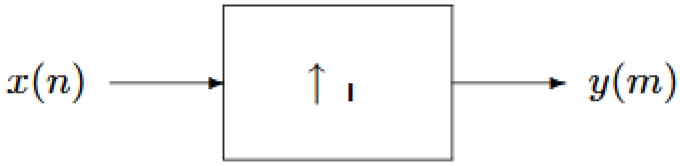
This equation relates the spectrum of $x(n)$ and $y(n)$.

Interpolation (up-sampling):

Interpolation is a process of increasing the sampling rate by a integer factor of I .The interpolation pads I-1 new samples between successive values of the signal.Interpolartion process increases the sampling rate, it is symbolically Represented by an up-arrow (↑I).It is also called upsampling. The interpolation converts the input sequence into a sequence $y(n)$.

$$y(n) = x\left(\frac{n}{I}\right)$$

The upsampler is represented by the following diagram.



Problems:

1. For the given data sequence $x(n) = \{1,4,6,8,10\}$, find the output sequence which is upsampled version of $x(n)$ by $(i)2$ $(ii)3$.

   Soln:

   $x(n) = \{1,4,6,8,10\}$

$(i) y(n) = (\uparrow 2)x(n)$

$$= \{1,0,4,0,6,0,8,0,10\}$$

$(ii) y(n) = (\uparrow 3)x(n)$

$$= \{1,0,0,4,0,0,6,0,0,8,0,0,10\}$$

INTERPOLATION BY FACTOR I:

Interpolation by a factor I,means to increase the sampling rate by a factor I.It is also called upsampling by I.

If the sampling frequency of input signal is f_x, then it is increased by I.Thus the sampling frequency of the output signal is $f_y = I f_x$which is obtained by adding I-1 zeros between successive values of $x(n)$.

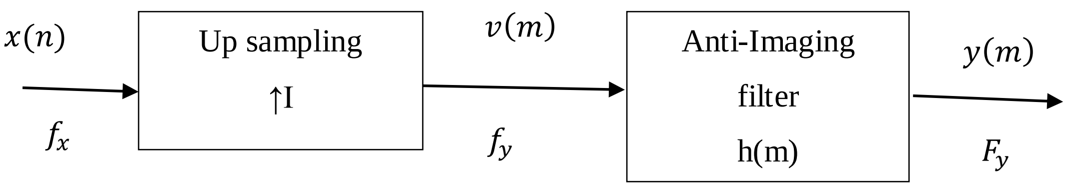


Fig: Interpolation by factor I

Derivation of Interpolation Equation:

The output $y(m)$ of the anti-imaging filter is given as the convolution of $v(m) * h(m)$.

$$y(m) = v(m) * h(m).$$

$$y(m) = \sum_{k=-\infty}^{\infty} h(m-k)v(k)$$

Relation between the spectrums of $x(n) and\ y(n)$.

Step (1):

$$v(m) = \begin{cases} x\left(\frac{m}{I}\right) & , m = 0 \pm I, \pm 2I \ldots \\ o\ , otherwise & \end{cases}$$

This means $v(m)$ is non-zero only at integer multiples of I.

Step (2):

Taking Z-Transform of $v(m)$, we get,

$$V(Z) = \sum_{m=-\infty}^{\infty} v(m)z^{-m}$$

$$V(Z) = \sum_{m=-\infty}^{\infty} x\left(\frac{m}{I}\right) z^{-m}$$

Sub $p = \frac{m}{I}$

When $m = -\infty\ , p = -\infty$

$$m = \infty\ , p = \infty$$

$$V(Z) = \sum_{p=-\infty}^{\infty} x(p)z^{-pI}$$

Changing dummy variable $p\ to\ m$.

$$V(Z) = \sum_{m=-\infty}^{\infty} x(m)(Z^{I})^{-m}$$

$$V(Z) = X(Z^{I})$$

Step (3):

To find out the spectrum sub $z = e^{j\omega_y}$, whereω_y represents the frequency in the output spectrum.

$$V\left(e^{j\omega_y}\right) = X\left(e^{j\omega_y I}\right)$$

Normally equ can be denoted as

$$V(\omega_y) = X(\omega_y I)$$

<u>Step (4):</u>

The input and output frequency are related by the equ

$$\omega_y = \frac{\omega_x}{I}$$

Frequency components of $x(n)$ are in the range $0 \leq \omega_y \leq \frac{\pi}{I}$ are unique, the images of $X(\omega)$ above $\omega_y = \frac{\pi}{I}$ should be rejected by passing the sequence $v(m)$through a low pass filter with frequency response $H_I(\omega_y)$.

$$H_I(\omega_y) = \begin{cases} C \text{ },0 \leq |\omega_y| \leq \dfrac{\pi}{I} \\ 0, else \end{cases}$$

Where C is a scale factor required to properly normalize the output sequence $y(m)$.

Consequently ,the output spectrum is obtained as follows.

$$y(m) = h(m) * v(m)$$

$$Y(Z) = H(Z)V(Z)$$

Let $z = e^{j\omega_y}$

$$Y(e^{j\omega_y}) = H(e^{j\omega_y})V(e^{j\omega_y})$$

$$Y(\omega_y) = H(\omega_y)V(\omega_y)$$

$$Y(\omega_y) = \begin{cases} CX(\omega_y I) \text{ },0 \leq |\omega_y| \leq \dfrac{\pi}{I} \\ 0, else \end{cases}$$

To find out the output in time domain ,we have to take inverse Fourier transform.

$$y(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega_y) e^{j\omega_y m} \, d\omega_y$$

$$y(m) = \frac{1}{2\pi} \int_{-\frac{\pi}{I}}^{\frac{\pi}{I}} CX(\omega_y I) e^{j\omega_y m} \, d\omega_y$$

Let $m = 0$

$$y(0) = \frac{1}{2\pi} \int_{-\frac{\pi}{I}}^{\frac{\pi}{I}} CX(\omega_y I) \, d\omega_y$$

$$= \frac{C}{2\pi} \int_{-\frac{\pi}{I}}^{\frac{\pi}{I}} X(\omega_y I)\, d\omega_y$$

We know that,

$$\omega_y = \frac{\omega_x}{I}$$

$$\omega_x = \omega_y I$$

$$If\ \omega_y = \frac{-\pi}{I} \qquad , \omega_x = -\pi$$

$$\omega_y = \frac{+\pi}{I} \qquad , \omega_x = +\pi$$

$$d\omega_x = d\omega_y I$$

$$d\omega_y = \frac{d\omega_x}{I}$$

$$y(0) = \frac{C}{2\pi} \int_{-\pi}^{\pi} X(\omega_x) \frac{d\omega_x}{I}$$

$$y(0) = \frac{C}{2\pi} \cdot \frac{1}{I} \int_{-\pi}^{\pi} X(\omega_x) d\omega_x$$

$$= \frac{C}{I} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega_x) d\omega_x \right]$$

$$y(0) = \frac{C}{I} x(0)$$

$$Let\ x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega_x) d\omega_x$$

When $C = I$ then we get desired normalization factor.

Finally we indicate the output sequence $y(m)$ can be expressed as a convolution of the sequence $v(n)$witn the unit sample response $h(n)$ of the low pass filter.

Thus,

$$y(m) = v(m) * h(m)$$

$$y(m) = \sum_{k=-\infty}^{\infty} v(k) h(m-k)$$

$Replace\ K\ by\ KI\ \ and\ V(KI) = x(K)$

$$y(m) = \sum_{k=-\infty}^{\infty} v(kI)h(m - kI)$$

$$y(m) = \sum_{k=-\infty}^{\infty} x(k)h(m - kI)$$

SAMPLING RATE CONVERSION BY A RATIONAL FACTOR $I/_D$:

Let as consider the general case of sampling rate conversion by a rational factor $I/_D$ .We can achieve this sampling rate conversion by first performing interpolation by the factor I and then decimating the output of the interpolator by the factor. Inother words, the sampling rate conversion by the rational factor $I/_D$is accomplished by cascading an interpolator with a decimator.

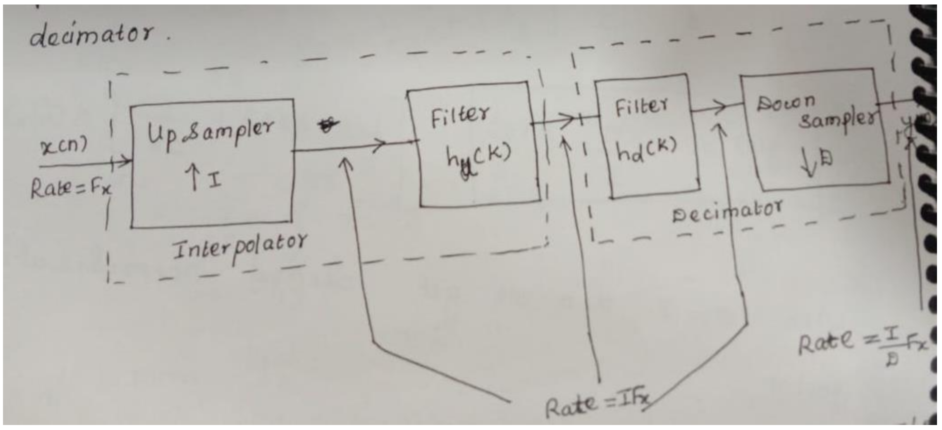


Fig: method for sampling rate conversion by factor I/D

The interpolation by a factor I is obtained first to increase the sampling rate to IF_x.

The output of the interpolator is then decimated by a factor D, so that the final output rate is

$$F_y = \frac{IF_x}{D}$$

In the above figure, observe that there is a cascade of two low pass filters. The overall cut off frequency will be minimum of the two cut off frequencies.

The frequency response of the anti-imaging filter is given as,

$$H_u(\omega) = \begin{cases} C & , \frac{-\pi}{I} \le \omega \le \frac{\pi}{I} \\ & 0, else \end{cases}$$

The scaling factor $C = I$ for a desired normalization.

$$H_u(\omega) = \begin{cases} I & , \frac{-\pi}{I} \leq \omega \leq \frac{\pi}{I} \\ & 0, else \end{cases}$$

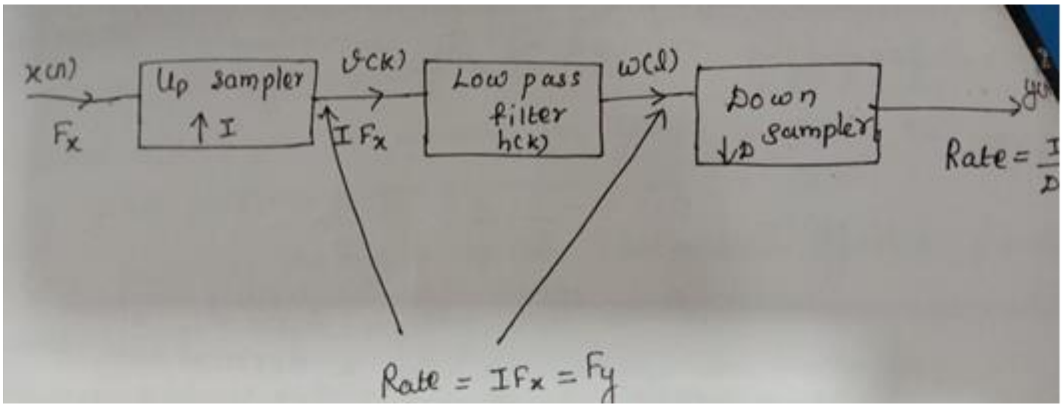
The frequency response of anti-imaging filter is given as,

$$H_d(\omega) = \begin{cases} 1 & , \frac{-\pi}{D} \leq \omega \leq \frac{\pi}{D} \\ & 0, else \end{cases}$$

The overall cascading effect of low pass filter will have a cutoff frequency which is minimum of $\frac{\pi}{I}$ and $\frac{\pi}{D}$. Hence we can write the frequency response of the combined filter as

$$H(\omega) = \begin{cases} I, |\omega| \leq min\left(\frac{-\pi}{D}, \frac{\pi}{I}\right) \\ 0, other\ wise \end{cases}$$

Thus single filter can be used ,as designed by above equation. Then the block diagram can be modified as follows.



<u>Derivation for output $y(m)$.</u>

The output of low pass filter is given by

$$w(l) = v(k)h(k)$$

$$w(l) = \sum_{k=-\infty}^{\infty} v(k)\, h(l-k)$$

$k = Ik$

$$w(l) = \sum_{k=-\infty}^{\infty} v(kI)\, h(l-kI)$$

Let $V(kI) = x(k)$

$$w(l) = \sum_{k=-\infty}^{\infty} x(k)\, h(l-kI)$$

The output of the down sampler

$$y(m) = w(mD)$$

Sub $l = mD$ in the above equ,

$$w(mD) = \sum_{k=-\infty}^{\infty} x(k)h(mD - kI)$$

$$y(m) = \sum_{k=-\infty}^{\infty} x(k)h(mD - kI)$$

This is the equation for output sequence.

The frequency domain relationship can be obtained by combining the results of the interpolation and decimation processes. Thus the spectrum of the up sample is

$$Y(\omega_y) = \begin{cases} CX(\omega_y I), 0 \le |\omega_y| \le \frac{\pi}{I} \\ 0, else \end{cases}$$

$where\ C = I$

$$Y(\omega_y) = \begin{cases} IX(\omega_y I), 0 \le |\omega_y| \le \frac{\pi}{I} \\ 0, else \end{cases}$$

Here $w(l)$ is the output of the linear filter

$$W(\omega_v) = \begin{cases} IX(\omega_v I), 0 \le |\omega_v| \le min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0, else \end{cases} ----(1)$$

The spectrum of the down sampler is

$$Y(\omega_y) = \frac{1}{D} X\left(\frac{\omega_y}{D}\right) \ for\ 0 \le |\omega_y| \le \pi$$

Here $w(l)$ is the input of the down sampler

$$Y(\omega_y) = \frac{1}{D} W\left(\frac{\omega_y}{D}\right) ----(2)$$

The input and output frequency are related by the equ,

$$\omega_x = \frac{\omega_y}{D}$$

Here

$$\omega_x = \omega_v$$

$$\omega_v = \frac{\omega_y}{D} --- (3)$$

(2) implies

$$Y(\omega_y) = \frac{1}{D} W(\omega_v)$$

$$Y(\omega_y) = \frac{1}{D} \begin{cases} IX(\omega_v I), 0 \le |\omega_v| \le min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0, else \end{cases}$$

$$\omega_v = \frac{\omega_y}{D}$$

$$Y(\omega_y) = \begin{cases} \frac{I}{D} X\left(\frac{I}{D}\omega_y\right), 0 \le \left|\frac{\omega_y}{D}\right| \le min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0, else \end{cases}$$

$$Y(\omega_y) = \begin{cases} \frac{I}{D} X\left(\frac{I}{D}\omega_y\right), 0 \le |\omega_y| \le min\left(\pi, \frac{\pi D}{I}\right) \\ 0, else \end{cases}$$

This gives the spectrum of the output sequence.