2.4 LAPLACE TRANSFORM

It is used to transform a time domain to complex frequency domain signal(s-domain).

Two Sided Laplace transform (or) Bilateral Laplace transform

Let $x(t)$ be a continuous time signal defined for all values of t. Let $X(S)$ be Laplace transform of $x(t)$.

$$L\{x(t)\} = X(S) = \int_{-\infty}^{\infty} x(t)e^{-St}\,dt$$

One sided Laplace transform (or) Unilateral Laplace transform

Let $x(t)$ be a continuous time signal defined for $t \geq 0$ (ie If $x(t)$ is causal) then,

$$L\{x(t)\} = X(S) = \int_{0}^{\infty} x(t)e^{-St}\,dt$$

Inverse Laplace transform

The S-domain signal $X(S)$ can be transformed to time domain signal x(t) by using inverse Laplace transform. The inverse Laplace transform of X(S) is defined as,

$$L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(S)e^{st}\,ds$$

Existence of Laplace transform

The necessary and sufficient conditions for the existence of Laplace transform are

- $x(t)$ should be continuous in the given closed interval
- $x(t)$ must be absolutely intergrable.

$$i.e., X(S)\ exists\ only\ if\ \int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|dt < \infty$$

PROBLEMS:

1. Find unilateral Laplace transform for the following signals.

   (i) $\boldsymbol{x(t) = \delta(t)}$

$$X(S) = \int_0^{\infty} x(t)e^{-st}dt = \int_0^{\infty} \delta(t)e^{-st}dt = e^{-s(0)} = 1$$

   (ii) $\boldsymbol{x(t) = u(t)}$

$$X(S) = \int_0^{\infty} x(t)e^{-st}dt = \int_0^{\infty} u(t)e^{-st}dt$$

$$= \int_0^{\infty} 1.e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s}$$

2. Find the Laplace Transform of $x(t) = e^{at}u(t)$.

   Solution:

$$X(S) = L[e^{at}u(t)]$$

$$= \int_0^{\infty} e^{at}\,e^{-st}dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = \frac{1}{s-a}$$

3. Determine initial value and final value of the following signal $X(S) = \frac{1}{s(s+2)}$.

   Solution:

   Initial value:

$$x(0) = \underset{s\to\infty}{\text{Lt}}\; SX(S) = \underset{s\to\infty}{\text{Lt}}\; s\frac{1}{s(s+2)} = \frac{1}{\infty} = 0$$

   Final Value:

$$x(\infty) = \underset{s\to 0}{\text{Lt}}\; SX(S) = \underset{s\to 0}{\text{Lt}}\; s\frac{1}{s(s+2)} = \frac{1}{2}$$