Find the Fourier cosine and sine transform of $f(x)=\begin{cases} x, & for\ 0<x<1 \\ 2-x, & for\ 1<x<2 \\ 0, & for\ x>2 \end{cases}$.

Solution:

Given $f(x)=\begin{cases} x, & for\ 0<x<1 \\ 2-x, & for\ 1<x<2 \\ 0, & for\ x>2 \end{cases}$

The Fourier Cosine transform of f(x) is

$$F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx\,.$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx\,dx+\int_1^2(2-x)\cos sx\,dx+\int_2^\infty 0\cos sx\,dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[(x)\left(\frac{\sin sx}{s}\right)-(1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1+\left[(2-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\frac{x\sin sx}{s}+\frac{\cos sx}{s^2}\right]_0^1+\left[(2-x)\left(\frac{\sin sx}{s}\right)-\frac{\cos sx}{s^2}\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\left(\frac{\sin s}{s}+\frac{\cos s}{s^2}\right)-\left(0+\frac{1}{s^2}\right)\right]+\left[\left(0-\frac{\cos 2s}{s^2}\right)-\left(\frac{\sin s}{s}-\frac{\cos s}{s^2}\right)\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s}+\frac{\cos s}{s^2}-\frac{1}{s^2}-\frac{\cos 2s}{s^2}-\frac{\sin s}{s}+\frac{\cos s}{s^2}\right]$$

$$F_c(s)=\sqrt{\frac{2}{\pi}}\left[\frac{2\cos s-\cos 2s-1}{s^2}\right]$$

The Fourier sine transform of f(x) is

$$F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin sx\,dx+\int_1^2(2-x)\sin sx\,dx+\int_2^\infty 0\sin sx\,dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[(x)\left(\frac{-\cos sx}{s}\right)-(1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^1+\left[(2-x)\left(\frac{-\cos sx}{s}\right)-(-1)\left(\frac{-\sin sx}{s^2}\right)\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\frac{-x\cos sx}{s}+\frac{\sin sx}{s^2}\right]_0^1+\left[-(2-x)\left(\frac{\cos sx}{s}\right)-\frac{\sin sx}{s^2}\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\left(-\frac{\cos s}{s}+\frac{\sin s}{s^2}\right)-(0)\right]+\left[\left(0-\frac{\sin 2s}{s^2}\right)-\left(-\frac{\cos s}{s}-\frac{\sin s}{s^2}\right)\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s}+\frac{\sin s}{s^2}-\frac{\sin 2s}{s^2}+\frac{\cos s}{s}+\frac{\sin s}{s^2}\right]$$

$$F_s(s)=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin s-\sin 2s}{s^2}\right]$$

Find Fourier transform of $e^{-a|x|}$ and hence deduce that

(a) $\int_0^\infty \frac{\cos xt}{a^2+t^2}dt = \frac{\pi}{2a}e^{-a|x|}$ **(b)** $F\left[xe^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(s^2+a^2\right)^2}$ **.**

The Fourier transform of f(x) is

$$\boxed{F\left[f(x)\right] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\ .}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\left(\cos sx + i\sin sx\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-a|x|}\cos sx\,dx + i\int_{-\infty}^{\infty} e^{-a|x|}\sin sx\,dx\right]$$

$\because e^{-a|x|}\sin sx$ is an *odd* fn $\therefore \int_{-\infty}^{\infty} e^{-a|x|}\sin sx\,dx = 0$

$$= \frac{1}{\sqrt{2\pi}}2\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$F(s) = F\left[e^{-a|x|}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$ $\because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$ here $a = a; b = s$

Deduction (a):

By inverse Fourier transform of *F(s)* is

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left[\frac{a}{a^2+s^2}\right]\left(\cos sx - i\sin sx\right)ds$$

$$= \frac{a}{\pi}\left[\int_{-\infty}^{\infty}\left[\frac{1}{a^2+s^2}\right]\left(\cos sx\right)ds - ia\int_{-\infty}^{\infty}\left[\frac{1}{a^2+s^2}\right]\left(\sin sx\right)ds\right]$$

$f(x) = \frac{2a}{\pi}\int_0^\infty\left[\frac{1}{a^2+s^2}\right]\cos sx\,ds$ $\because \left(\frac{1}{a^2+s^2}\right)\left(\sin sx\right)$ is an odd function

$$\int_0^\infty\left(\frac{1}{a^2+s^2}\right)\cos sx\,ds = \frac{\pi}{2a}f(x)$$

$$\boxed{\int_0^\infty \frac{\cos sx}{a^2+s^2}ds = \frac{\pi}{2a}e^{-a|x|}}$$

Put *s=t*

$$\boxed{\int_0^\infty \frac{\cos tx}{t^2+a^2}dt = \frac{\pi}{2a}e^{-a|x|}}$$

Deduction (b):

By Property

$$F\left[x\, f(x)\right] = -i\frac{d}{ds}\left[F(s)\right]$$

$$F\left[xe^{-a|x|}\right] = -i\frac{d}{ds}F\left(e^{-a|x|}\right)$$

$$= -i\frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right)$$

$$= -ia\sqrt{\frac{2}{\pi}}\left(\frac{-1}{\left(a^2+s^2\right)^2}(0+2s)\right) = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(a^2+s^2\right)^2}$$

$$\boxed{F\left[xe^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}}\left(\frac{2as}{\left(s^2+a^2\right)^2}\right)}$$

Find the Fourier sine and cosine transform of $e^{-ax}, a > 0$ **and deduce that**

i) $\int_0^\infty \frac{s}{s^2+a^2}\sin sx\, dx = \frac{\pi}{2}e^{-ax}$.

ii) $\int_0^\infty \frac{1}{s^2+a^2}\cos sx\, dx = \frac{\pi}{2a}e^{-ax}$

Solution:

The Fourier sine transform of f(x) is

$$F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx .$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\, dx$$

$$\boxed{F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2} \text{ here } a = a; b = s$$

The Fourier cosine transform of f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx .$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\, dx$$

$$\boxed{F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2} \text{ here } a = a; b = s$$

The inverse Fourier sine transform of $F_s(s)$ is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sin sx\, dx$$

$$= \frac{2}{\pi}\int_0^\infty \left[\frac{s}{a^2+s^2}\right]\sin sx\, dx$$

$$\int_0^\infty \left[\frac{s}{a^2+s^2}\right]\sin sx\,dx = \frac{\pi}{2}\,f(x)$$

$$\boxed{\int_0^\infty \left[\frac{s}{a^2+s^2}\right]\sin sx\,dx = \frac{\pi}{2}e^{-ax}}$$

The inverse Fourier Cosine transform of $F_c(s)$ is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\cos sx\,dx$$

$$= \frac{2a}{\pi}\int_0^\infty \left[\frac{1}{a^2+s^2}\right]\cos sx\,dx$$

$$\int_0^\infty \left[\frac{a}{a^2+s^2}\right]\cos sx\,dx = \frac{\pi}{2}\,f(x)$$

$$\boxed{\int_0^\infty \left[\frac{a}{a^2+s^2}\right]\cos sx\,dx = \frac{\pi}{2a}e^{-ax}}$$

Find the Fourier sine and cosine transform of $e^{-ax}, a>0$ **and hence find** $F_c\left[xe^{-ax}\right]$ and $F_s\left[xe^{-ax}\right]$.

Solution:

The Fourier sine transform f(x) is

$$F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2} \quad \text{here } a=a; b=s$$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\boxed{F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \quad \text{here } a=a; b=s$$

We know that

i) $F_s\left[xf(x)\right] = -\frac{d}{ds}\left\{F_c\left[f(x)\right]\right\} = -\frac{d}{ds}\left[F_c(s)\right]$

$$F_s\left[xe^{-ax}\right] = -\frac{d}{ds}\left\{F_c\left[e^{-ax}\right]\right\} = -\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\right\}$$

$$= -a\sqrt{\frac{2}{\pi}}\,\frac{d}{ds}\left\{\frac{1}{a^2+s^2}\right\}$$

$$= -a\sqrt{\frac{2}{\pi}}\left[\frac{-1}{\left(a^2+s^2\right)^2}(0+2s)\right]$$

$$\boxed{F_s\left[xe^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{\left(a^2+s^2\right)^2}\right]}$$

ii) $F_c\left[xf(x)\right] = \frac{d}{ds}\left\{F_s\left[f(x)\right]\right\} = \frac{d}{ds}\left[F_s(s)\right]$

$$F_s\left[xe^{-ax}\right] = \frac{d}{ds}\left\{F_c\left[e^{-ax}\right]\right\} = \frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{\left(a^2+s^2\right)(1) - s(0+2s)}{\left(a^2+s^2\right)^2}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{a^2+s^2-2s^2}{\left(a^2+s^2\right)^2}\right\}$$

$$\boxed{F_s\left[xe^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{\left(a^2+s^2\right)^2}\right]}$$

Find the Fourier sine transform of $\frac{e^{-ax}}{x}, a > 0$ **and hence find** $F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$.

Solution:

The Fourier sine transform of f(x) is

$$F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

Taking diff. on both sides w.r.to s

$$\frac{d}{ds}\left\{F_s\left[\frac{e^{-ax}}{x}\right]\right\} = \frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}\left(\sin sx\right)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{\cancel{x}}\left(\cos sx\right)\cancel{x}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\frac{d}{ds}\left\{F_s\left[\frac{e^{-ax}}{x}\right]\right\} = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$$

Integrating on on both sides w.r.to s

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int\left[\frac{a}{a^2+s^2}\right]ds$$

$$\boxed{F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)} \qquad \because \int\frac{a}{x^2+a^2}dx=\tan^{-1}\left(\frac{x}{a}\right)$$

Similarly, $F_s\left[\frac{e^{-bx}}{x}\right]=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$

Deduction:

$$F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]=F_s\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$$

$$=F_s\left[\frac{e^{-ax}}{x}\right]-F_s\left[\frac{e^{-bx}}{x}\right]$$

$$=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)-\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$$

$$\boxed{F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]=\sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{s}{a}\right)-\tan^{-1}\left(\frac{s}{b}\right)\right]}$$

Find the Fourier cosine transform of $\frac{e^{-ax}}{x}, a>0$ **and hence find** $F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

Solution:

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\cos sx\,dx$$

Taking diff. on both sides w.r.to s

$$\frac{d}{ds}\left\{F_c\left[\frac{e^{-ax}}{x}\right]\right\}=\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\cos sx\,dx\right\}$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\frac{\partial}{\partial s}\left(\cos sx\right)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{\cancel{x}}\left(-\sin sx\right)\cancel{x}\,dx$$

$$=-\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\frac{d}{ds}\left\{F_c\left[\frac{e^{-ax}}{x}\right]\right\}=-\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$$

Integrating on on both sides w.r.to s

$$F_c\left[\frac{e^{-ax}}{x}\right]=-\sqrt{\frac{2}{\pi}}\int\left[\frac{s}{a^2+s^2}\right]ds$$

$$=-\sqrt{\frac{2}{\pi}}\int\left[\frac{s}{a^2+s^2}\right]ds$$

$$= -\sqrt{\frac{2}{\pi}}\frac{1}{2}\int\left[\frac{2s}{a^2+s^2}\right]ds$$

$$= -\sqrt{\frac{2}{\pi}}\frac{1}{2}\log\left(s^2+a^2\right) \qquad \because \int\frac{f'(x)}{f(x)}dx = \log\left[f(x)\right]$$

$$= -\frac{1}{\sqrt{2\pi}}\log\left(s^2+a^2\right)$$

$$= \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right)$$

$$\boxed{F_c\left[\frac{e^{-ax}}{x}\right] = \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right)}$$

Similarly $F_c\left[\frac{e^{-bx}}{x}\right] = \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+b^2}\right)$

Deduction:

$$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$$

$$= F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right) - \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+b^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$\boxed{F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)}$$

13. **Using Parseval's identity evaluate the following integrals.**

1) $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ **2)** $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx$, where $a > 0$.

Solution:

Assume $f(x) = e^{-ax}$

The Fourier sine transform f(x) is

$$F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2} \text{ here } a=a; b=s$$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos x\,dx$$

$$\boxed{F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \text{ here } a = a; b = s$$

(i) The Parseval's identity for Fourier cosine transform is

$$\int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |f(x)|^2 dx$$

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\right)^2 ds = \int_0^\infty \left(e^{-ax}\right)^2 dx$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2}\,ds = \int_0^\infty e^{-2ax}dx$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2}\,ds = \frac{\pi}{2a^2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2}\,ds = \frac{-\pi}{4a^3}\left[e^{-\infty} - e^{-0}\right]$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2}\,ds = \frac{-\pi}{4a^3}\left[0-1\right] \qquad \because e^{-\infty} = 0;\ e^{-0} = 1$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2}\,ds = \frac{\pi}{4a^3}$$

Put s=x we get

$$\boxed{\int_0^\infty \frac{1}{\left(a^2+x^2\right)^2}\,dx = \frac{\pi}{4a^3}}$$

(ii) The Parseval's identity for Fourier sine transform is

$$\int_0^\infty |F_s(s)|^2\,ds = \int_0^\infty |f(x)|^2 dx$$

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\right)^2 ds = \int_0^\infty \left(e^{-ax}\right)^2 dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds = \int_0^\infty e^{-2ax}dx$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds = \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds = \frac{-\pi}{4a}\left[e^{-\infty} - e^{-0}\right]$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds = \frac{-\pi}{4a}\left[0-1\right] \qquad \because e^{-\infty} = 0;\ e^{-0} = 1$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds = \frac{\pi}{4a}$$

$$\boxed{\int_0^\infty \frac{x^2}{\left(a^2+x^2\right)^2}\,dx = \frac{\pi}{4a}}$$

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