4.3 Transformation of analog to digital filters

The objective of impulse invariant transformation is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter.The main idea behind this techniques is to preserve the frequency response characteristics of the analog filter.It can be stated that the frequency response of digital filter will be identical with the frequency response of of the corrosponding analog filter if the sampling time period T is selected sufficiently small to minimize the effect of aliasing.

Let h(t) = Impulse response of analog filter

Transfer function of analog filter,H(s) = L{ h(t)}

$$H(s) = \sum_{i=1}^{N} \frac{A_i}{s+p_i} = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \ldots\ldots\ldots + \frac{A_N}{s+p_N}$$

Let h(n) = Impulse response of the digital filter

The impulse response of the digital filter is obtained by uniformly sampling the impulse response of the analog filter.

$h(n) = h(t)|_{t=nT}$

$h(n) = h(t)|_{t=nT} = h(nT) = \sum_{i=1}^{N} A_i \; e^{-p_i nT}$ u(nT)

$= A_1 e^{-p_1 nT}$u(nT) + $A_1 e^{-p_2 nT}$u(nT) + $A_1 e^{-p_3 nT}$u(nT)+….+ $A_1 e^{-p_N nT}$u(nT)

On taking the Z transform of equation we get

$$H(z) = Z\{h(n)\} == A_1 \frac{1}{1-e^{-p_1 T} z^{-1}} + A_2 \frac{1}{1-e^{-p_2 T} z^{-1}} + \ldots\ldots + A_N \frac{1}{1-e^{-p_N T} z^{-1}}$$

Comparing the expression of H(s) and H(z) we can say that

$$\frac{1}{s+p_i} \rightarrow \frac{1}{1-e^{-p_i T} z^{-1}} \quad \ldots\ldots\ldots\ldots.(1)$$

By impulse invariant transformation, where T is the sampling period.

Relation between Analog and digital filter poles in Impulse Invariant transformation

The analog poles are given by the roots of the term$(s + p_i)$, for i= 1,2,3…N.The digital poles are given by the roots of the term $(1 - e^{-p_i T} z^{-1})$,for i=1,2,3….N.From equation (1) we can say that the analog poles at s=-p_i is transformed into a digital pole at z=$e^{-p_i T}$

The following observations can be made

1. If the analog pole s_i lie on Left Half of s-plane the corresponding digital pole z_I lie inside the unit circle in z plane.

2. If the analog pole s_i lie on Imaginary axis of s-plane the corresponding digital pole z_i lie on the unit circle in z plane.

3. If the analog pole s_i lie on the Right Half of s-plane the corresponding digital pole z_i lie on the outside the unit circle in z plane.

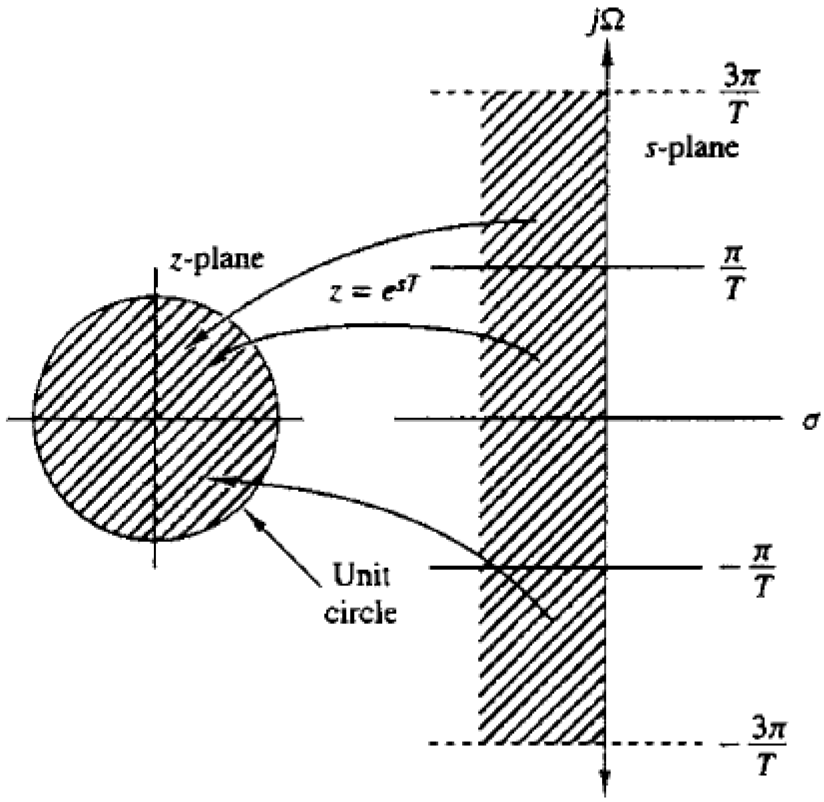

Fig.The mapping of s plane into z plane

The stability of the filter is related to the location of the pole.

Example -1

For the analog transfer function ,H(s)=$\frac{2}{S^2+3s+2}$,determine H(z) using impulse invariant transformation if (a) T=1 second and (b) T=0.1 second

Given that:

$$H(s)=\frac{2}{S^2+3s+2}$$

$$=\frac{2}{(s+1)(s+2)}$$

By partial fraction expansion technique we can write

H(s) = $\frac{2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$

A = $\frac{2}{(s+1)(s+2)} \times (s+1)\,|_{s=-1}$

= $\frac{2}{1} = 2$

B = $\frac{2}{(s+1)(s+2)} \times (s+2)\,|_{s=-2} = \frac{2}{-2+1}$ =-2

H(s) = $\frac{2}{(s+1)} + \frac{-2}{(s+2)}$

= $\frac{2}{(s+1)} - \frac{2}{(s+2)}$

By impulse invariant transformation we know that

$\frac{1}{s+p_i} \quad \rightarrow \quad \frac{1}{1-e^{-p_i T}z^{-1}}$

H (z) = $\frac{2}{1-e^{-p_1 T}z^{-1}} + \frac{-2}{1-e^{-p_2 T}z^{-1}}$ where p1 = 1 and p2 = 2

H (z) = $\frac{2}{1-e^{-T}z^{-1}} + \frac{-2}{1-e^{-2T}z^{-1}}$

a)When T= 1 second

H (z) = $\frac{2}{1-e^{-T}z^{-1}} + \frac{-2}{1-e^{-2T}z^{-1}}$

H (z) = $\frac{2}{1-0.3679\,z^{-1}} + \frac{-2}{1-0.1353z^{-1}}$

= $\frac{2}{1-0.3679\,z^{-1}} - \frac{2}{1-0.1353z^{-1}}$

=$\frac{2(1-0.1353z^{-1}) - 2(1-0.3679\,z^{-1})}{(1-0.3679\,z^{-1})(1-0.1353z^{-1})}$

=$\frac{2-0.2706z^{-1}-2+0.7358\,z^{-1}}{1-0.1353z^{-1}-0.367z^{-1}+0.0498\,z^{-2}}$

= $\frac{2-0.2706z^{-1}-2+0.7358\,z^{-1}}{1-0.1353z^{-1}-0.367z^{-1}+0.0498\,z^{-2}}$

= $\frac{0.4652\,z^{-1}}{1-0.503z^{-1}+0.0498\,z^{-2}}$

$$\mathbf{H(z)=} \frac{\mathbf{0.4652\,z^{-1}}}{\mathbf{1-0.503z^{-1}+0.0498\,z^{-2}}}$$

b)When T=0.1 second

H (z) = $\frac{2}{1-e^{-T}z^{-1}} + \frac{-2}{1-e^{-2T}z^{-1}}$

= $\frac{2}{1-e^{-0.1}z^{-1}} + \frac{-2}{1-e^{-0.2}z^{-1}}$

$$=\frac{2}{1-0.9048z^{-1}}+\frac{-2}{1-0.8187z^{-1}}$$

$$=\frac{(1-0.8187z^{-1})2-2(1-0.9048z^{-1})}{(1-0.9048z^{-1})(1-0.8187z^{-1})}$$

$$=\frac{2-1.637z^{-1}-2+1.8096z^{-1}}{1-0.8187z^{-1}-0.9048z^{-1}+0.7408\,z^{-2}}$$

$$=\frac{0.1722\,z^{-1}}{1-1.7235z^{-1}+0.7408\,z^{-2}}$$

$$\mathbf{H(z)}=\frac{\mathbf{0.1722\,z^{-1}}}{\mathbf{1-1.7235z^{-1}+0.7408\,z^{-2}}}$$

IIR FILTERS DESIGN-BILINEAR TRANSFORMATION

The bilinear transformation is a conformal mapping that transforms the imaginary axis of s-plane into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. In this mapping all points in the left half of s plane are mapped inside the unit circle in the Z plane and all points in the right half of s-plane are mapped outside unit circle in the Z-plane.

The bilinear transformation can be linked to the trapezoidal formula for numerical integration.

By the bilinear transformation

$$S\,Y(s) \rightarrow \frac{2\,(1-z^{-1}\,)}{T\;(1+z^{-1}\,)}\quad Y\,(z)$$

Here in the s domain transfer function H(s) is transformed into Z domain transfer function by substituting s by the term $\frac{2\,(1-z^{-1}\,)}{T\;(1+z^{-1}\,)}$.

Relation between Analog and Digital poles in bilinear transformation:

The mapping of s domain function into z domain function by bilinear transformation is a one to one mapping, that is for every point in z plane ,there is exactly one corresponding point in s plane and vice versa.

The variable "s" represents a point on s plane and "z" is the corresponding point in z plane.

The following observations can be made

1. The point s_i lie on the left half of s plane the corresponding point in z plane will lie inside the unit circle in the z plane.

2. The point s_i lie on the right half of s plane the corresponding point in z plane will lie outside the unit circle in the z plane.

3. The point s_i lie on the imaginary axis of s plane the corresponding point in z plane will lie on the unit circle in the z plane.

The above discussions are applicable for mapping poles and zeros from s plane to z plane. The stability of the filter is associated with location of poles. We know that for a stable analog filter the poles should lie on the left half of s plane.

Frequency Warping

The change in properties when using Bilinear Transformation is referred to as Frequency Warping. The non-linear relationship between Ω and ω results in a distortion of the frequency axis. The spectral representation of frequency using Bilinear Transformation differs from the usual representation

Pre-Warping

Frequency warping follows a known pattern, and there is a known relationship between the warped frequency and the known frequency. We can use a technique called prewarping to account for the nonlinearity, and produce a more faithful mapping. You can remove the warping problem using a simple technique.

Example-1:

For the analog transfer function, H(s) = $\frac{2}{s^2+3s+2}$, determine H(z) using bilinear transformation if a) T= 1 second b) T = 0.1 second

Given that, H(s) = $\frac{2}{s^2+3s+2}$

Put, s = $\frac{2\,(1-z^{-1}\,)}{T\;(1+z^{-1}\,)}$

H (z) = $\frac{2}{\left[\frac{2\,(1-z^{-1}\,)}{T\;(1+z^{-1}\,)}\right]^2+3\,\left[\frac{2\,(1-z^{-1}\,)}{T\;(1+z^{-1}\,)}\right]+2}$

$=\frac{2T^2(1+z^{-1}\,)^2}{4(1-z^{-1}\,)^2+\,6T\,((1-z^{-2}\,)+2T^2\,(1+z^{-1}\,)^2}$

When a) T=1 second

H (z)= $\frac{2(1+z^{-1}\,)^2}{4(1-z^{-1}\,)^2+\,6(1-z^{-2}\,)+2(1+z^{-1}\,)^2}$

$=\frac{2(1+2z^{-1}\,+z^{-2}\,)}{4(1-2z^{-1}\,+z^{-2}\,)+\,6(1-z^{-2}\,)+(1+2z^{-1}\,+z^{-2}\,)2}$

$=\frac{2+4z^{-1}\,+\,2z^{-2}}{12-4z^{-1}}$

$=\frac{2+4z^{-1}\,+\,2z^{-2}}{12(1-\frac{4}{12}z^{-1}\,)}$

$=\frac{0.1667+0.3333z^{-1}\,+0.1667z^{-2}}{1-0.333z^{-1}}$

$$\mathbf{H(z)} = \frac{\mathbf{0.1667+0.3333}z^{-1}\,\mathbf{+0.1667}z^{-2}}{\mathbf{1-0.333}z^{-1}}$$

When a) T=0.1 second

$$H\ (z) = \frac{2*0.1^2(1+z^{-1}\ )^2}{4(1-z^{-1}\ )^2 + 6*0.1\ ((1-z^{-2}\ )+2*0.1^2\ (1+z^{-1}\ )^2}$$

$$= \frac{0.02(1+2z^{-1}\ +z^{-2}\ )}{4(1-2z^{-1}\ +z^{-2}\ )+0.6(1-z^{-2}\ )+0.02(1+2z^{-1}\ +z^{-2})}$$

$$= \frac{0.02z^{-1}\ +0.04z^{-1}\ +0.02z^{-1}}{4.62-7.96z^{-1}\ +3.42z^{-2}}$$

$$= \frac{0.0043+0.0087z^{-1}\ +0.0043z^{-2}}{1-1.7229z^{-1}\ +0.7403z^{-2}}$$

$$\mathbf{H(z)} = \frac{\mathbf{0.0043+0.0087}\boldsymbol{z}^{-1}\ \mathbf{+0.0043}\boldsymbol{z}^{-2}}{\mathbf{1-1.7229}\boldsymbol{z}^{-1}\ \mathbf{+0.7403}\boldsymbol{z}^{-2}}$$

IIR FILTER DESIGN USING APPROXIMATION OF DERIVATIVES

In this method, the differential equation of analog filter is approximated by an equivalent difference equation of the digital filter. For the derivative $\frac{dy(t)}{dt}$ at $t = nT$, following substitution is made,

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT - T)}{T}$$

Here T is the sampling interval and $y(n) \equiv y(nT)$, hence above equation can be written as,

$$\frac{dy(t)}{dt} = \frac{y(n) - y(n-1)}{T} \qquad \text{... (2.3.4)}$$

The system function of the differentiator having output $\frac{dy(t)}{dt}$ is,

$$H(s) = s \qquad \text{... (2.3.5)}$$

The system function of the digital filter which produces output $\frac{y(n) - y(n-1)}{T}$ is

$$H(z) = \frac{1 - z^{-1}}{T} \qquad \text{... (2.3.6)}$$

Thus the analog domain to digital domain transformation can be obtained (from equation (2.3.4), equation (2.3.5) and equation (2.3.6)as

$$s = \frac{1 - z^{-1}}{T} \qquad \text{... (2.3.7)}$$

Similarly it can be shown that,

$$s^k = \left(\frac{1 - z^{-1}}{T}\right)^k \qquad \text{... (2.3.8)}$$

Here 'k' represents the order of the derivative. Thus the system function of the digital filter can be obtained from the system function of the analog filter by approximation of derivatives as follows.

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} \qquad \text{... (2.3.9)}$$

S-PLANE TO Z-PLANE MAPPING

From equation (2.3.6) we have,

$$z = \frac{1}{1-sT} \quad \text{... (2.3.10)}$$

We know that $s = \sigma + j\Omega$, hence above equation becomes,

$$z = \frac{1}{1-(\sigma+j\Omega)T} = \frac{1}{1-\sigma T - j\Omega T} = \frac{1-\sigma T + j\Omega T}{(1-\sigma T)^2 + (\Omega T)^2}$$

$$= \frac{1-\sigma T}{(1-\sigma T)^2 + (\Omega T)^2} + j\frac{\Omega T}{(1-\sigma T)^2 + (\Omega T)^2} \quad \text{... (2.3.11)}$$

Let us see how $j\Omega$ axis is mapped in z-plane. For this, substitute $\sigma = 0$ in above equation. Hence we get,

$$z = \frac{1}{1+(\Omega T)^2} + j\frac{\Omega T}{1+(\Omega T)^2} \quad \text{... (2.3.12)}$$

The above equation shows that complete $j\Omega$ axis ($-\infty$ *to* $+\infty$) is mapped on the circle of radius $1/2$ and center at $z = \frac{1}{2}$. This is shown in Fig. 2.3.1. This circle is inside the unit circle. From equation (2.3.11) it can be shown that left hand plane of $j\Omega$ axis maps inside

the circle of radius $\frac{1}{2}$ centred at $z = \frac{1}{2}$. And right hand plane of $j\Omega$ axis maps outside this circle. This is shown in Fig. 2.3.1. Thus a stable analog filter is converted to stable digital filter.

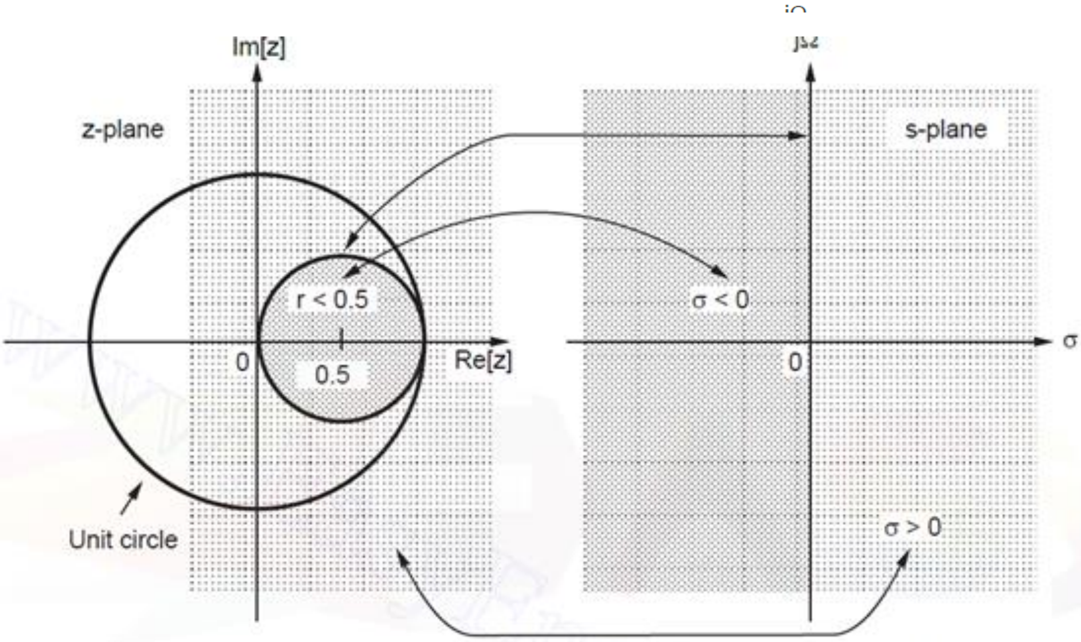

Fig. Mapping from *s*-plane to *z*-plane for s $= \frac{1-z^{-1}}{T}$
i.e. approximation of derivatives method