ROHINI cOLLEGE OF ENGINEERING \& TECHNOLOGY DEPARTMENT OF MATHEMATICS

## UNIT I - FOURIER SERIES

## Harmonic Analysis

The process of finding the Fourier series for a function given by numerical values is known as harmonic analysis.

$$
f(x)=\frac{a_{0}}{2}-+\sum_{n=1}^{\infty}\left(a_{n} \cos n x+b_{n} \operatorname{sinnx}\right) \text {, where }
$$

$\mathrm{ie}, \mathrm{f}(\mathrm{x})=(\mathrm{a} 0 / 2)+(\mathrm{a} 1 \cos \mathrm{x}+\mathrm{b} 1 \sin \mathrm{x})+(\mathrm{a} 2 \cos 2 \mathrm{x}+\mathrm{b} 2 \sin 2 \mathrm{x})+(\mathrm{a} 3 \cos 3 \mathrm{x}+\mathrm{b} 3 \sin 3 \mathrm{x})+-$ -----------...(1)

$$
\begin{aligned}
& \text { Here } \mathrm{a}_{0}=2[\text { mean values of } \mathrm{f}(\mathrm{x})]=-\quad-------- \\
& \text { n }
\end{aligned}
$$

n
\& $\quad b_{n}=2[$ mean values of $f(x) \operatorname{sinn} x]=-------------$
n
In (1), the term $(\operatorname{arcos} x+b 1 \sin x)$ is called the fundamental or first harmonic, the term $(a 2 \cos 2 \mathrm{x}+\mathrm{b} 2 \sin 2 \mathrm{x})$ is called the second harmonic and so on.

## Problem 1.

Compute the first three harmonics of the Fourier series of $f(x)$ given by the following table.

| $\mathrm{x}:$ | 0 | $\pi / 3$ | $2 \pi / 3$ | $\pi$ | $4 \pi / 3$ | $5 \pi / 3$ | $2 \pi$ |
| :--- | :--- | :--- | :--- | :---: | :--- | :--- | :--- |
| $\mathrm{f}(\mathrm{x}):$ | 1.0 | 1.4 | 1.9 | 1.7 | 1.5 | 1.2 | 1.0 |

We exclude the last point $\mathrm{x}=2 \pi$.
Let $\mathrm{f}(\mathrm{x})=(\mathrm{a} 0 / 2)+(\mathrm{a} 1 \cos \mathrm{x}+\mathrm{b} 1 \sin \mathrm{x})+(\mathrm{a} 2 \cos 2 \mathrm{x}+\mathrm{b} 2 \sin 2 \mathrm{x})+\ldots \ldots \ldots$.
To evaluate the coefficients, we form the following table.

| x | $\mathrm{f}(\mathrm{x})$ | $\cos \mathrm{x}$ | $\sin \mathrm{x}$ | $\cos 2 \mathrm{x}$ | $\sin 2 \mathrm{x}$ | $\cos 3 \mathrm{x}$ | $\sin 3 \mathrm{x}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 1.0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\pi / 3$ | 1.4 | 0.5 | 0.866 | -0.5 | 0.866 | -1 | 0 |
| $2 \pi / 3$ | 1.9 | -0.5 | 0.866 | -0.5 | -0.866 | 1 | 0 |
| $\pi$ | 1.7 | -1 | 0 | 1 | 0 | -1 | 0 |
| $4 \pi / 3$ | 1.5 | -0.5 | -0.866 | -0.5 | 0.866 | 1 | 0 |
| $5 \pi / 3$ | 1.2 | 0.5 | -0.866 | -0.5 | -0.866 | -1 | 0 |

$$
\therefore \mathrm{f}(\mathrm{x})=1.45-0.37 \cos \mathrm{x}+0.17 \sin \mathrm{x}-0.1 \cos 2 \mathrm{x}-0.06 \sin 2 \mathrm{x}+0.033 \cos 3 \mathrm{x}+\ldots
$$

## Problem 2

Obtain the first three coefficients in the Fourier cosine series for $y$, where $y$ is given in the following table:

$$
\begin{align*}
& \text { Now, } \mathrm{a}_{0}=\underline{2 \sum \mathrm{f}(\mathrm{x})}=\underline{2(1.0+1.4+1.9+1.7+1.5+1.2)}=2.9 \\
& 6  \tag{6}\\
& a_{1}=-----------\quad=-0.37 \\
& 6 \\
& 2 \sum \mathrm{f}(\mathrm{x}) \cos 2 \mathrm{x} \\
& a_{2}=-------------\quad=-1 \\
& 6 \\
& a_{3}=------------\quad=0.033 \\
& 6 \\
& 2 \sum \mathrm{f}(\mathrm{x}) \sin \mathrm{x} \\
& \mathrm{~b}_{1}=-------------=0.17 \\
& 6 \\
& b_{2}=--------------\quad=-0.06 \\
& 6 \\
& b_{3}=\frac{2 \sum f(x) \sin 3 x}{6}=0
\end{align*}
$$

| $\mathrm{x}:$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{y}:$ | 4 | 8 | 15 | 7 | 6 | 2 |

Taking the interval as $60^{\circ}$, we have

| $\theta:$ | $0^{\circ}$ | $60^{\circ}$ | $120^{\circ}$ | $180^{\circ}$ | $240^{\circ}$ | $300^{\circ}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{x}:$ | 0 | 1 | 2 | 3 | 4 | 5 |
| $\mathrm{y}:$ | 4 | 8 | 15 | 7 | 6 | 2 |

$\therefore$ Fourier cosine series in the interval $(0,2 \pi)$ is $\mathrm{y}=(\mathrm{a} 0 / 2)+\mathrm{a} \cos \theta+$ $a 2 \cos 2 \theta+\mathrm{a} \cos 3 \theta+\ldots .$.
To evaluate the coefficients, we form the following table.

| $\theta^{\circ}$ | $\cos \theta$ | $\cos 2 \theta$ | $\cos 3 \theta$ | y | $\mathrm{y} \cos \theta$ | $\mathrm{y} \cos 2 \theta$ | $\mathrm{y} \cos 3 \theta$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $0^{\circ}$ | 1 | 1 | 1 | 4 | 4 | 4 | 4 |
| $60^{\circ}$ | 0.5 | -0.5 | -1 | 8 | 4 | -4 | -8 |
| $120^{\circ}$ | -0.5 | -0.5 | 1 | 15 | -7.5 | -7.5 | 15 |
| $180^{\circ}$ | -1 | 1 | -1 | 7 | -7 | 7 | -7 |
| $240^{\circ}$ | -0.5 | -0.5 | 1 | 6 | -3 | -3 | 6 |
| $300^{\circ}$ | 0.5 | -0.5 | -1 | 2 | 1 | -1 | -2 |

Now, $\quad \mathrm{a} 0=2(42 / 6)=14$
a1 $=2(-8.5 / 6)=-2.8$
a2 $=2 \quad(-4.5 / 6)=$
a3 $=2(8 / 6)=2.7$
$\mathrm{y}=7-2.8 \cos \theta-1.5 \cos 2 \theta+2.7 \cos 3 \theta+\ldots .$.

## Problem 3

The values of $x$ and the corresponding values of $f(x)$ over a period $T$ are given below. Show that $f(x)=0.75+0.37 \cos \theta+1.004 \sin \theta$, where $\theta=(2 \pi x \quad) / T$

| $\mathrm{x}:$ | 0 | $\mathrm{~T} / 6$ | $\mathrm{~T} / 3$ | $\mathrm{~T} / 2$ | $2 \mathrm{~T} / 3$ | $5 \mathrm{~T} / 6$ | T |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{y}:$ | 1.98 | 1.30 | 1.05 | 1.30 | -0.88 | -0.25 | 1.98 |

We omit the last value since $f(x)$ at $x=0$ is known.
Here $\theta=2 \pi \mathrm{x} / \mathrm{T}$
When x varies from 0 to $\mathrm{T}, \theta$ varies from 0 to $2 \pi$ with $2 \pi / 6$. an incre

Let $\mathrm{f}(\mathrm{x})=\mathrm{F}(\theta)=(\mathrm{a} 0 / 2)+\mathrm{a} 1 \cos \theta+\mathrm{b}_{1} \sin \theta$.
To evaluate the coefficients, we form the following table.

| $\theta$ | y | $\cos \theta$ | $\sin \theta$ | $\mathrm{y} \cos \theta$ | $\mathrm{y} \sin \theta$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 1.98 | 1.0 | 0 | 1.98 | 0 |
| $\pi / 3$ | 1.30 | 0.5 | 0.866 | 0.65 | 1.1258 |
| $2 \pi / 3$ | 1.05 | -0.5 | 0.866 | -0.525 | 0.9093 |
| $\Pi$ | 1.30 | -1 | 0 | -1.3 | 0 |
| $4 \pi / 3$ | -0.88 | -0.5 | -0.866 | 0.44 | 0.762 |
| $5 \pi / 3$ | -0.25 | 0.5 | -0.866 | -0.125 | 0.2165 |
|  | 4.6 |  |  | 1.12 | 3.013 |

Now, $\mathrm{a} 0=2\left(\sum \mathrm{f}(\mathrm{x}) / 6\right)=1.5$

$$
\begin{aligned}
& \mathrm{a} 1=2(1.12 / 6)=0.37 \\
& \mathrm{a} 2=2(3.013 / 6)=1.004
\end{aligned}
$$

Therefore, $\mathrm{f}(\mathrm{x})=0.75+0.37 \cos \theta+1.004 \sin \theta$

