INNER PRODUCT

Definition: Let V be a vector space over a ficld F, An inner product on V is a function from $V \times V \rightarrow F$ that assigns, to ever ordered pair of vectors x and y in V, a scalar in F, denoted by $\langle x, y\rangle$ such that for all $x, y, z \in V$ and scalar $\alpha \in F$ the following axioms hold:

$$I_1: \langle x, x\rangle > 0 \text{ if } x \neq 0$$
$$I_2: \langle x+z, y\rangle = \langle x, y\rangle + (z, y)$$
$$I_3: \langle \alpha x, y\rangle = \alpha(x, y)$$

$I_4: \overline{\langle x, y\rangle} = \langle y, x\rangle$, where the bar denotes the complex conjucation.

Note:

For real numbers i.e., $F = R$, the complex conjugate of a number is itself. Then I_3 reduces to

$$\langle x, y\rangle = \langle y, x\rangle$$

Properties of inner product:

If V is an inner product space, then for $x, y, z \in V$ and scalar $\alpha \in F$ the following statements are true.

(i) $\langle x, 0\rangle = (0, x) = 0$

(ii) $(x, x) = 0$ if and only if $x = 0$

(iii) $\langle x, \alpha y\rangle = \bar{\alpha}(x, y)$

(iv) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

(v) $(x, y) = \langle x, z\rangle$ for all $x \in V$, then $y = z$.

Proof:

$$\begin{aligned} \text{(i) } \langle 0, x\rangle &= \langle 0+0, x\rangle \\ &= \langle 0, x\rangle + \langle 0, x\rangle = 0 \\ \therefore \langle x, 0\rangle &= \langle \bar{0}, x\rangle = \bar{0} = 0 \end{aligned}$$

(ii) $(x, x) = 0$ if and only if $x = 0$

Let $x = 0$. Then $\langle x, x\rangle = \langle 0,0\rangle = 0$

We know that $\langle x, x\rangle > 0$ if $x \neq 0$

Obviously $\langle x, x\rangle = 0$ if and only if $x = 0$.

(iii)

$$
\begin{aligned}
(x, a y) &= \overline{\langle a y, x\rangle} \\
&= \overline{\alpha(y, x)} \\
&= \bar{\alpha}(y, x) \\
&= \bar{\alpha}(x, y) \\
\therefore \overline{\langle a x, y)} = \bar{\alpha}(x, y) &= \overline{\langle y+z, x\rangle} \\
\text{(iv)}(x, y+z) &= \langle\overline{\langle y, x\rangle}+\overline{\langle z, x\rangle}+\langle x, z\rangle \\
&=\therefore (x, y+z)=\langle x, y\rangle+\langle x, z\rangle
\end{aligned}
$$

(iv) $(x, y+z)=\overline{(y+z, x)}$

$=\overline{\langle y, x\rangle}+\overline{\langle z, x\rangle}=\langle x, y)+(x, z) \therefore (x, y+z)=\langle x, y\rangle+(x, z)$

(v) Assume $\langle x, y\rangle=\langle x, z\rangle$ ... (1), for all $x \in V$

$$
\begin{aligned}
\text{Consider } (x, y-z) &= \langle x, y\rangle-\langle x, z\rangle \\
&= \langle x, y\rangle-\langle x, y\rangle[\text{ From } (i v)] \\
&= 0 \ldots (2)
\end{aligned}
$$

Take $x=y-z$, we get,

$$
\begin{aligned}
& \langle y-z, y-z\rangle=0 \\
& \Rightarrow y-z=0
\end{aligned}
$$

$$\Rightarrow y=z$$

If $x \neq y$, then from (2), we get

Either $x=0$ or $y-z=0$

$\therefore y=z$

Definition: Inner product space

A vector space endowed with a specific inner product is called product space.

Standard inner product of F^n

Let $x, y \in F^n$. Then $x = (a_1, a_2, \ldots, a_n)$ and $y=(b_1, b_2, \ldots, b_n)$. The inner

$$\langle x, y\rangle = a_1 \overline{b_1}+a_2 \overline{y b_2}+\cdots+a_n \overline{b_n}$$

is called standard inner product on F^n.

Standard inner product of R^n

Let $x, y \in R^n$. Then $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$. The inner $\langle x, y\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ is called standard inner product on R^n.

3.1.1. PROBLEMS UNDER INNER PRODUCT SPACE

1. Let $x = (a_1, a_2, \dots, a_n), y = (b_1, b_2, \dots, b_n) \in F^n$. Define inner product $\langle x, y\rangle = a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n}$. Verify F^n is an inner space.

Sol: Let $x, y, z \in V$ and $\alpha \in F$.

Let $x = (a_1, a_2, \dots, a_n); y = (b_1, b_2, \dots, b_n)$ and $z = (c_1, c_2, \dots, c_n)$

Given $\langle x, y\rangle = a_1\bar{b}_1 + a_2\overline{b_2} + \cdots + a_n\overline{b_n}$

$I_1: \langle x, x\rangle > 0$ if $x \neq 0$

$$\begin{aligned}\langle x, x\rangle &= a_1\overline{a_1} + a_2\overline{a_2} + \cdots + a_n\overline{a_n}\\ &= |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 > 0 \ [\because a_i \neq 0 \text{ for some } i]\\ \therefore \langle x, x\rangle &> 0 \text{ if } x \neq 0\end{aligned}$$

$\langle x, x\rangle = a_1\overline{a_1} + a_2\overline{a_2} + \cdots + a_n\overline{a_n}$

$= |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 > 0$

$\therefore (x, x) > 0$ if $x \neq 0$

$I_2: \langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

$x + z = (a_1, a_2, \dots, a_n) + (c_1, c_2, \cdots, c_n) = (a_1 + c_1, a_2 + c_2, \dots, a_n + c_n)$

$(x + z, y) = (a_1 + c_1)\bar{b}_1 + (a_2 + c_2)\overline{b_2} + \cdots + (a_n + c_n)\overline{b_n} = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n + c_1\bar{b}_1 + c_2\bar{b}_2 + \cdots + c_1^2 + x, x)$

$$\begin{aligned}&= a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n + c_1\bar{b}_1 + c_2\overline{b_2} + \cdots + c_n\overline{b_n}\\ &= \langle x, y\rangle + \langle z, y\rangle\end{aligned}$$

$\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

$I_3: \langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

We have $x = (a_1, a_2, \dots, a_n)$.

$\therefore ax = (aa_1, aa_2 + \cdots, \alpha a_n)(ax, y) = aa_1\overline{b_1} + \alpha a_2\overline{b_2} + \cdots + \alpha a_n\overline{b_n}$

$$= a(a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n}) = a\langle x, y\rangle \therefore \langle ax, y\rangle = a\langle x, y\rangle$$

$I_4: \langle x, y\rangle = \langle y, x\rangle \langle x, y\rangle = a_1\bar{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n}\widehat{\langle x, y\rangle}$

$$= \overline{a_1\bar{b}_1 + a_2\overline{b_2} + \cdots + a_n\overline{b_n}} = \overline{a_1}b_1 + \overline{a_2}b_2 + \cdots + \overline{a_n}b_n$$

$$= b_1\overline{a_1} + b_2\overline{a_2} + \cdots + b_n\overline{a_n} = (y, x) \therefore \overline{\langle x, y\rangle} = \langle y, x\rangle$$

2. Consider the vector space R^n. Prove that R^n is an inner product space with inner product $\langle x, y\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ where $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$.

Sol: Let $x, y, z \in V$ and $\alpha \in F$.

Let $x = (a_1, a_2, \dots, a_n); y = (b_1, b_2, \dots, b_n)$ and $z = (c_1, c_2, \dots, c_n)$

Given $\langle x, y\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$I_1: (x, x) > 0$ if $x \neq 0$

$\langle x, x\rangle = a_1a_1 + a_2a_2 + \cdots + a_na_n$

$= a_1^2 + a_2^2 + \cdots + a_n^2 > 0$ [$\because a_i \neq 0$ for some i]

$\therefore (x, x) > 0$ if $x \neq 0$

$I_2: (x + z, y) = \langle x, y\rangle + \langle z, y\rangle x + z = (a_1, a_2, \dots, a_n) + (c_1, c_2, \dots, c_n)$

$$= (a_1 + c_1, a_2 + c_2, \dots, a_n + c_n)(x + z, y)$$

$$= (a_1 + c_1)b_1 + (a_2 + c_2)b_2 + \cdots + (a_n + c_n)b_n$$

$$= a_1b_1 + a_2b_2 + \cdots + a_nb_n + c_1b_1 + c_2b_2 + \cdots + c_nb_n$$

$$= \langle x, y\rangle + \langle z, y\rangle(x + z, y) = \langle x, y\rangle + (z, y)I_3: \langle ax, y\rangle = \alpha(x, y)$$

We have $x = (a_1, a_2, \dots, a_n)$.

$\alpha x = (\mathrm{a}a_1, \mathrm{a}a_2, \dots, \mathrm{a}a_n)$ $\begin{aligned}\langle \alpha x, y\rangle &= aa_1b_1 + aa_2b_2 + \cdots + aa_nb_1 \\ &= a(a_1b_1 + a_2b_2 + \cdots + a_nb_n)\end{aligned} = a\langle x, y\rangle$

$$\therefore \langle \alpha x, y\rangle = a\langle x, y\rangle$$

$I_4: \overline{\langle x, y\rangle} = \langle y, x\rangle$

$\langle x, y\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$$\begin{aligned}\overline{\langle x, y\rangle} &= \overline{a_1b_1 + a_2b_2 + \cdots + a_nb_n} \\ &= a_1b_1 + a_2b_2 + \cdots + a_nb_n \\ &= b_1a_1 + b_2a_2 + \cdots + b_na_n \\ &= \langle y, x\rangle \\ \therefore \overline{\langle x, y\rangle} &= \langle y, x\rangle\end{aligned}$$

Hence R^n is an inner product space.

3. Prove that R^2 is an inmer product space with an inner product defined by $\langle x, y\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$ where $x = (a_1, a_2)$; $y = (b_1, b_2)$.

Sol; Let $x, y, z \in R^2$ and $\alpha \in F$

Lat $x = (a_1, a_2); y = (b_1, b_2)$ and $z = (c_1, c_2)$

Given $\langle x, y\rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$

I_1 ; $\langle x, x\rangle > 0$ if $x \neq 0$

$\langle x, x\rangle = a_1a_1 - a_2a_1 - a_1a_2 + 2a_2a_2 = a_1^2 - 2a_1a_2 + 2a_2^2$

$= a_1^2 - 2a_1a_2 + a_2^2 + a_2^2$

$= (a_1 - a_2)^2 + a_2^2 > 0$ [$\because a_1 \neq 0$ or $a_2 \neq 0$]

$\therefore \langle x, x\rangle > 0$ if $x \neq 0$

I_2; $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle x + z = (a_1, a_2) + (c_1, c_2)$

$= (a_1 + c_1, a_2 + c_2)\langle x + Z, y\rangle$

$= (a_1 + c_1)b_1 - (a_2 + c_2)b_1 - (a_1 + c_1)b_2 + 2(a_2 + c_2)b_2$

$= a_1b_1 + c_1b_1 - a_2b_1 - c_2b_1 - a_1b_2 - c_1b_2 + 2a_2b_2 + 2c_2b_2$

$= a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2 + c_1b_1 - c_2b_1 - c_1b_2 + 2c_2b_2$

$= \langle x, y\rangle + \langle z, y\rangle \therefore \langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

I_3: $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

We have $x = (a_1, a_2)$

$\therefore \alpha x = (\alpha a_1, \alpha a_2)$

$$\begin{aligned}\langle \alpha x, y\rangle &= \alpha a_1b_1 - \alpha a_2b_1 - \alpha a_1b_2 + 2\alpha a_2b_2\\ &= \alpha(a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2)\\ &= \alpha\langle x, y\rangle\end{aligned}$$

$\therefore \langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

I_4: $\overline{\langle x, y\rangle} = \langle y, x\rangle$

$$\begin{aligned}\langle x, y\rangle &= a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2\\ \overline{\langle x, y\rangle} &= \overline{a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2}\\ &= a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2\\ &= b_1a_1 - b_2a_2 - a_2b_1 + 2b_2a_2\\ &= \langle y, x\rangle\end{aligned}$$

$\therefore \overline{\langle x, y\rangle} = \langle y, x\rangle$

4. Let V be the set of all real functions defined on the clo interval $[0,1]$. The inner product on V is defined by $\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)$ Prove that $V(R)$ is an inner product space.

Sol:

Let $f, g, h \in V$ and $\alpha \in F$.

Hence R^2 is an inner product space with the given inner product.

$$\text{Given } \langle f,g\rangle = \int_0^1 f(t)g(t)dt$$

$$I_1: \langle f,f\rangle > 0 \text{ if } f \neq 0$$

$$\langle f,f\rangle = \int_{-1}^{1} f(t)f(t)dt$$

$$= \int_{-1}^{1} [f(t)]^2 dt > 0$$

$$\therefore \langle f,f\rangle > 0 \text{ if } f \neq 0$$

$$I_2: \langle f+h,g\rangle = \langle f,g\rangle + \langle h,g\rangle$$

$$\langle f+h,g\rangle = \int_{-1}^{1} [f(t)+h(t)]g(t)dt$$

$$= \int_{-1}^{1} f(t)g(t)\Big) dt + \int_{-1}^{1} h(t)g(t)dt$$

$$= \langle f,g\rangle + \langle h,g\rangle$$

$$\therefore \langle f+h,g\rangle = \langle f,g\rangle + \langle h,g\rangle$$

$$I_3: \langle \alpha f,g\rangle = \alpha\langle f,g\rangle$$

$$\langle \alpha f,g\rangle = \int_{-1}^{1} (\alpha f)(t)g(t)dt$$

$$= \alpha \int_{-1}^{1} f(t)g(t)dt$$
$$= \alpha\langle f, g\rangle$$
$$\therefore \langle(\alpha f, g) = \alpha\langle f, g\rangle$$
$$I_4: \overline{\langle f, g\rangle} = \langle g, f\rangle$$
$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$$
$$\overline{\langle f, g\rangle} = \int_{-1}^{1} f(t)g(t)dt$$
$$= \int_{-1}^{1} f(t)g(t)dt$$
$$= \int_{-1}^{1} g(t)f(t)dt$$
$$= \langle g, f\rangle$$
$$\therefore \overline{(f, g)} = \langle g, f\rangle$$

Therefore $V(R)$ is an inner product space.

5. Let H be the vector space of all continuous complex value functions on $[0, 1]$. Show that V is a complex inner product space with is product $\langle f, g\rangle = \frac{1}{3\pi}\int_0^1 f(t)\overline{g(t)}dt$.

Sol:

Let $f, g, h \in V$ and $a \in F$.

$$\text{Given } \langle f, g\rangle = \frac{1}{2\pi}\int_0^1 f(t)g(t)dt$$
$$l_1: \langle f, f\rangle > 0 \text{ if } f \neq 0$$
$$\langle f, f\rangle > 0 \text{ for } f \neq 0$$

$\langle f, f\rangle = \dfrac{1}{2\pi}\int_0^1 f(t)\overline{f(t)}dt$

$= \frac{1}{2\pi}\int_0^1 |f(t)|^2 dt > 0$

$\therefore (f, f) > 0$ if $f \neq 0$

$I_2: (f+h,g) = \langle f,g\rangle + \langle h,g\rangle \langle f+h,g\rangle = \frac{1}{2\pi}\int_0^1 (f+h)(t)\overline{g(t)}dt$

$$= \frac{1}{2\pi}\int_0^1 - [f(t)+h(t)]\overline{g(t)}dt$$

$$= \frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}dt + \frac{1}{2\pi}\int_0^1 h(t)\overline{g(t)}dt = \langle f,g\rangle + \langle h,g\rangle$$

$$\therefore \langle f+h,g\rangle = \langle f,g\rangle + \langle h,g\rangle I_3: \langle \alpha f,g\rangle = \alpha\langle f,g\rangle\langle \alpha f,g\rangle$$

$$= \frac{1}{2\pi}\int_0^1 (\alpha f)(t)\overline{g(t)}dt = a\frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}dt = \alpha\langle f,g\rangle$$

$$\therefore \langle \alpha f,g\rangle = \alpha\langle f,g\rangle I_4: \overline{(f,g)} = (g,f)(f,g)$$

$$= \frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}dt\overline{(f,g)} = \overline{\frac{1}{2\pi}\int_0^1 f(t)\overline{g(t)}dt}$$

Therefore $V(C)$ is an inner product space.

3.1.2. NORM OF A VECTOR

Definition

Let V be an inner product space and let $x \in V$ then norm or length of x is $\| x \|$ and is defined by $\| x \| = \sqrt{\langle x,x\rangle}$

9. Find the norm of the following vectors in $V_3(R)$ with, inner product: $(a)(1,1,1), (b)(1,2,3), (c)(3,-4,0), (d)(4x+5y)$ where $x = (1,-1,0)$ and $y = (1,2,3)$

Sol:

Let x=(a_1, a_2, a_3) ; y=$(b_1, b_2, b_3) \in V_3$ (R)

$$The standard inner product space is$$

$$\langle x,y\rangle = \langle x,y\rangle = a_1b_1 + a_2b_2 + a_3b_3$$
$$\therefore (x,x) = a_1^2 + a_2^2 + a_3^3$$

(a)Let $x = (1,1,1)$

$$\begin{aligned} \| x \|^2 &= \langle x, x \rangle \\ &= 1^2 + 1^2 + 1^2 \\ &= 3 \\ \Rightarrow \| x \| &= \sqrt{3} \end{aligned}$$

(b) Let $x = (1,2,3)$

$$\begin{aligned} \| x \|^2 &= \langle x, x \rangle \\ &= 1^2 + 2^2 + 3^2 \\ &= 14 \\ \Rightarrow \| x \| &= \sqrt{14} \end{aligned}$$

(c) Let $x = (3, -4, 0)$

$$\begin{aligned} \| x \|^2 &= 3^2 + (-4)^2 + 0^2 \\ &= 9 + 16 \\ &= 25 \\ \Rightarrow \| x \| &= 5 \end{aligned}$$

(d) Let $u = 4x + 5y$

$$\begin{aligned} &= 4(1, -1, 0) + 5(1,2,3) \\ &= (4, -4, 0) + (5,10,15) \\ &= (9,6,15) \\ \| u \|^2 &= \langle u, u \rangle \\ &= 9^2 + 6^2 + 15^2 \\ &= 342 \\ \Rightarrow \| u \| &= \sqrt{342} \end{aligned}$$

10. Find the norm of the following vectors in Euclidean space R^3 with standard inner product $(a)\, u = (2, 1, -1), (b)\, v = \left(\frac{1}{2}, \frac{2}{3}, -\frac{1}{4}\right)$

Sol:

(a) Let $u = (2,1,-1)$

$\| u \|^2 = 2^2 + 1^2 + (-1)^2$

$=6$

$\| u \| = \sqrt{6}$

(b) Let $v = \left(\frac{1}{2}, \frac{2}{3}, -\frac{1}{4}\right)$

$\| v \|^2 = 6^2 + 8^2 + (-3)^2$

=109

$\| v \| = \sqrt{109}$

11. Find the norm of the following vectors in F^3 with standard inner product: $x = (1+i, 2, i), y = (3i, 2+3i, 4)$. Find (a) $\| x \|$, (b) $\| y \|$, (c) $\| x + y \|$, (d) $\langle x, y \rangle$

Sol: Let $x, y, z \in F^3$

Let $x = (a_1, a_2, a_3); y = (b_1, b_2, b_3)$

$$\langle x, y \rangle = a_1\bar{b}_1 + a_2\overline{b_2} + a_3\overline{b_3}$$
$$\langle x, x \rangle = |a_1|^2 + |a_2|^2 + |a_3|^2$$

Then

Sol:

(a) $\| x \|^2 = \langle x, x \rangle$
$= |1+i|^2 + |2|^2 + |i|^2$
$= 1^2 + 1^2 + 2^2 + 1^2$
$= 7$
$\| x \| = \sqrt{7}$

(b) $\| y \|^2 = \langle y, y \rangle$
$= |3i|^2 + |2+3i|^2 + |4|^2$
$= 3^2 + 2^2 + 3^2 + 4^2$
$= 9 + 4 + 9 + 16$
$= 38$
$\| y \| = \sqrt{38}$

(c) $x + y = (1+i, 2, i) + (3i, 2+3i, 4)$
$= (1+4i, 4+3i, 4+i)$
$\| x + y \|^2 = |1+4i|^2 + |4+3i|^2 + |4+i|^2$
$= 1^2 + 4^2 + 4^2 + 3^2 + 4^2 + 1^2$
$= 59$
$\| x + y \| = \sqrt{59}$

(d) $\langle x, y \rangle = \langle (1+i, 2, i), (3i, 2+3i, 4) \rangle$
$= (1+i)(\bar{3}i) + 2(2+3i) + i4$
$= (1+i)(-3i) + 2(2-3i) + 4i$
$= -3i + 3 + 4 - 6i + 4i$
$= 7 - 5i$

12. Let V be an vector space of polynomials with the inner product given by $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Let $f(t) = t + 2$ and $g(t) = t^2 - 2t - 3$ find (i)

$\langle \boldsymbol{f}, \boldsymbol{g} \rangle$ **(ii)** $\| \boldsymbol{f} \|$.

Sol:

$$\text{Let}\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

(i)
$$= \int_0^1 (t+2)(t^2-2t-3)dt$$

$$= \int_0^1 (t^3-2t^2-3t+2t^2-4t-6)dt$$

$$= \int_0^1 (t^3-7t-6)dt$$

$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$= -\frac{37}{4}$$

ii)

$$\| f \|^2 = \langle f, f\rangle$$

$$= \int_0^1 [f(t)]^2 dt$$

$$= \int_0^1 (t+2)^2 dt$$

$$= \int_0^1 (t^2+4t+4)dt$$

$$= \left[\frac{t^3}{3} + \frac{4t^2}{2} + 4t\right]_0^1$$

$$= \frac{1}{3} + 2 + 4$$

$$= \frac{19}{3}$$

$$\| f \| = \frac{\sqrt{19}}{\sqrt{3}}$$

13. For any non-zero vector, $x \in V$. prove that $y = \frac{x}{\|x\|}$ is a vector such that $\| y \| = 1$.

Sol: Consider

$$\begin{aligned} \langle y, y\rangle &= \left\langle \frac{x}{\| x \|}, \frac{x}{\| x \|} \right\rangle \\ &= \frac{1}{\| x \|} \cdot \frac{1}{\| x \|} \langle x, x\rangle \\ \langle y, y\rangle &= \frac{1}{\| x \|^2} \| x \|^2 \\ \| y \|^2 &= 1 \\ \| y \| &= 1 \end{aligned}$$

Theorem 3.1: In an inner product space V,

(i) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$

(ii) $\| \alpha x \| = |\alpha||x||$

Proof:

(i)

$$\begin{aligned} \| x \| &= \sqrt{\langle x, x\rangle} \\ \| x \|^2 &= \langle x, x\rangle \geq 0 \\ \| x \|^2 &\geq 0 \\ \| x \| &\geq 0 \end{aligned}$$

Also $(x, x) \geq 0$ if and only if $x = 0$

Therefore $\| x \|^2 = 0$ if and only if $x = 0$

(ii)

$$\begin{aligned} \| \alpha x \|^2 &= \langle \alpha x, \alpha x\rangle \\ &= \alpha\langle x, \alpha x\rangle \\ &= \alpha\bar{\alpha}\langle x, x\rangle \\ &= |\alpha|^2 \| x \|^2 \\ \| \alpha x \| &= |\alpha| \| x \| \end{aligned}$$

Theorem 3.2: [Schwarz's inequality]

For any two vectors x and y in an inner product space V,

$$|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| \leq \| \boldsymbol{x} \| \| \boldsymbol{y} \|$$

Proof:

If $x = 0$, then $\| x \| = 0$.
$\therefore \| x \| \| y \| = 0 \ldots$ (1)
Also $\langle x, y \rangle = \langle 0, y \rangle = 0$
$\therefore |\langle x, y \rangle| = 0 \ldots$ (2).

From (1) and (2)

$$|\langle x, y \rangle| = \| x \| \| y \|$$

So the result is true.

Let $x \neq 0$. Then $\| x \| > 0$

Therefore $\frac{1}{\|x\|}$ is a positive number

Consider the vector

$$w = y - \frac{\langle y, x \rangle}{\| x \|^2} x$$

$$\langle w, w \rangle = \left\langle y - \frac{\langle y, x \rangle}{\| x \|^2} x, y - \frac{\langle y, x \rangle}{\| x \|^2} x \right\rangle$$

$$= \langle y, y \rangle - \left\langle y \left[\frac{(y, x)}{\| x \|^2} ; \right\rangle - \left\langle \frac{\langle y, x \rangle}{\| x \|^2} x, y \right\rangle + \left\langle \frac{\langle y, x \rangle}{\| x \|^2} x, \frac{\langle y, x \rangle}{\| x \|^2} x \right\rangle$$

$$= \| y \|^2 - \frac{\text{Inner Product}}{\|x\|^2} \langle y, x \rangle - \frac{\langle y, x \rangle}{\|x\|^2} \langle x, y \rangle + \frac{\langle y, x \rangle \langle y, x \rangle}{\|x\|^4} \langle x, x \rangle$$

$$= \| y \|^2 - \frac{\langle x, y \rangle \langle \langle x, y \rangle}{\|x\|^2} - \frac{\frac{\langle x, y \rangle}{\langle x, y \rangle}}{\|x\|^2} + \frac{\overline{\langle x, y \rangle \rangle} \langle x, y \rangle \|x\|^2}{\|x\|^4}$$

$$= \| y \|^2 - \frac{|\langle x, y \rangle|^2}{\| x \|^2} - \frac{|\langle x, y \rangle|^2}{\| x \|^2} + \frac{|\langle x, y \rangle|^2}{\| x \|^2} [\because z\bar{z} = |z|^2]$$

$$(\text{w, w}) = \| y \|^2 - \frac{|\langle x, y \rangle|^2}{\| x \|^2}$$

$$\therefore \| y \|^2 - \frac{|\langle x, y \rangle|^2}{\| x \|^2} \geq 0$$

$$\| x \|^2 \| y \|^2 - |\langle x, y \rangle|^2 \geq 0$$

$\| x \|^2 \| y \|^2 \geq |\langle x, y \rangle|^2$ $|\langle x, y \rangle|^2 \leq \| x \|^2 \| y \|^2$

: $|\langle x, y \rangle| \leq \| x \| \| y \|$

Theorem 3.3: [Triangle inequality]

For any two vectors x and y in an inner product space V,

$$\| x+y \| \leq \| x \| + \| y \| .$$

Proof:

Using the norm of vectors we have

$$\begin{aligned}
| x+y \|^2 &= \langle x+y, x+y \rangle \\
&= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle \\
&= \| x \|^2 + \langle x,y \rangle + \overline{\langle x,y \rangle} + \| y \|^2 \\
&= \| x \|^2 + 2\text{Re}\langle x,y \rangle + \| y \|^2 \; [\because z+\bar{z} = 2\text{Re}(z)] \\
&\leq \| x \|^2 + 2|\langle x,y \rangle| + \| y \|^2 \; [\because \text{Re}(z) \leq |z|] \\
&\leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \; [\text{ By Shwarz's inequivality }] \\
&\leq (\| x \| + \| y \|)^2 \\
| x+y \|^2 &\leq (\| x \| + \| y \|)^2 \\
\| x+y \| &\leq \| x \| + \| y \|.
\end{aligned}$$

Theorem 3.4 : [Parallelogram law]

For any two vectors x and y in an inner product space V,

$\| x+y \|^2 + \| x-y \|^2 = 2(\| x \|^2 + \| y \|^2)$. What does this equation state about parallelograms in R^2 ?

Proof:

$$\begin{aligned}
\| x+y \|^2 &= \langle x+y, x+y \rangle \\
&= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle \\
&= \| x \|^2 + \langle x,y \rangle + \bar{x}, y + \| y \|^2 \\
&= \| x \|^2 + 2\text{Re}\langle x,y \rangle + \| y \|^2 \; \ldots (1)
\end{aligned}$$

and

$$\begin{aligned}
\| x-y \|^2 &= \langle x-y, x-y \rangle \\
&= \langle x,x \rangle - \langle x,y \rangle - \langle y,x \rangle + \langle y,y \rangle \\
&= \| x \|^2 - [\langle x,y \rangle + \langle x,y \rangle] + \| y \|^2 \\
&= \| x \|^2 - 2\text{Re}\langle x,y \rangle + \| y \|^2 \; \ldots (2)
\end{aligned}$$

$(1) + (2) \Rightarrow$

$$\| x+y \|^2 + \| x-y \|^2 = 2(\| x \|^2 + 2 \| y \|^2) \ldots (3)$$

Let $0ABC$ be a parallelogram with sides of length $0A = \| x \|$ and $0C = \| y \|$ in R^2. Therefore the length of the hypotenuses of $0ABC$ are $AC = \| x + y \|$ and $OB = \| x - y \|$

(3) $\Rightarrow 0B^2 + AC^2 = 0A^2 + AB^2 + BC^2 + CA^2 [\because |0A| = |BC|, |AB| = |C0|]$

Therefore sum of the squares of the two diagonals is equal to the sum of squares of four sides.