

Joint probability distribution function for continuous random variables X & Y

The joint probability distribution function of a two dimension as random variables

(X, Y) is denoted by $F_{XY}(x, y)$ and is given by

$$f_{XY}(x, y) \geq 0 \text{ and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$$

Properties of joint Distribution Functions

1. $F(-\infty, y) = 0 = F(x, \infty)$ and $F(-\infty, \infty) = 1$

2. $P(a_1 < X < b_1, a_2 < Y < b_2) = F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2)$

Marginal probability density function

The marginal probability density function of the two random variables X and Y are defined as follows

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy \text{ (Marginal pdf of X)}$$

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx \text{ (Marginal pdf of Y)}$$

Problems under Marginal Density function

1. The bivariate random variable X and Y has the pdf $f(x, y) =$

$Kx^2(8 - y), x < y < 2x, 0 \leq x \leq 2$. Find the value of K.

Solution:

We know that if $f(x, y)$ is a pdf then $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$

$$\Rightarrow \int_0^2 \int_x^2 K x^2 (8 - y) dx dy = 1$$

$$\Rightarrow K \int_0^2 x^2 \left(8y - \frac{y^2}{2} \right)_x^{2x} dx = 1$$

$$\Rightarrow K \int_0^2 x^2 \left(16x - \frac{4x^2}{2} - 8x + \frac{x^2}{2} \right)_x^{2x} dx = 1$$

$$\Rightarrow K \int_0^2 \left(16x^3 - 2x^4 - 8x^3 + \frac{x^4}{2} \right)_x^{2x} dx = 1$$

$$\Rightarrow K \int_0^2 \left(8x^3 - \frac{3x^4}{2} \right)_x^{2x} dx = 1$$

$$\Rightarrow K \left[\frac{8x^4}{4} - \frac{3x^5}{2 \cdot 5} \right]_0^2 = 1$$

$$\Rightarrow K \left[32 - \frac{48}{5} \right] = 1$$

$$\Rightarrow K \left(\frac{112}{5} \right) = 1 \Rightarrow K = \frac{5}{112}$$

2. The joint pdf of R.V X and Y is given by $f(x, y) = Kxye^{-(x^2+y^2)}$, $x >$

$0, y > 0$. Find the value of K and prove also that X and Y are independent.

Solution:

We know that if $f(x, y)$ is a p.d.f, then $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$

Here $f(x, y) = Kxye^{-(x^2+y^2)}$, $x > 0, y > 0$

We know that $\int_0^{\infty} \int_0^{\infty} Kxye^{-(x^2+y^2)} dy dx = 1$

$$\Rightarrow \int_0^{\infty} \int_0^{\infty} Kxye^{-x^2} e^{-y^2} dydx = 1$$

$$\Rightarrow K \int_0^{\infty} ye^{-y^2} dy \int_0^{\infty} xe^{-x^2} dx = 1$$

$$\Rightarrow K \frac{1}{2} \frac{1}{2} = 1 \Rightarrow K = 4$$

To prove X and Y are independent we have to prove that $f(x).f(y) = f(x, y)$

$$\begin{aligned} \text{Now, } f(x) &= f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \\ &= \int_0^{\infty} Kxye^{-(x^2+y^2)} dy \\ &= 4xe^{-x^2} \int_0^{\infty} ye^{-y^2} dy \\ &= 4xe^{-x^2} \left(\frac{1}{2}\right) \\ &= 2xe^{-x^2}, x > 0 \end{aligned}$$

Similarly $f(y) = f_Y(y) = 2ye^{-y^2}, y > 0$

$$\begin{aligned} \text{Now, } f(x).f(y) &= 2xe^{-x^2} 2ye^{-y^2} \\ &= 4xye^{-x^2} e^{-y^2} \\ &= 4xye^{-(x^2+y^2)} \\ &= f(x, y) \end{aligned}$$

\therefore X and Y are independent.

3. The joint probability density function of a two dimensional random variable

(X,Y) is given by $f(x, y) = xy^2 + \frac{x^2}{8}, 0 \leq x \leq 2, 0 \leq y \leq 1$ Compute

(i) $P(X > 1/Y < \frac{1}{2})$ (ii) $P(Y < \frac{1}{2}/X > 1)$ (iii) $P(X < Y)$ (iv) $P(X + Y \leq 1)$

Solution:

Given $f(x, y) = xy^2 + \frac{x^2}{8}, 0 \leq x \leq 2, 0 \leq y \leq 1$

$$\begin{aligned} \text{Now } P\left(X > 1/Y < \frac{1}{2}\right) &= \frac{P(X > 1, Y < \frac{1}{2})}{P(Y < \frac{1}{2})} \\ &= \int_0^{\frac{1}{2}} \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx dy \\ &= \int_0^{\frac{1}{2}} \left(\frac{x^2 y^2}{2} + \frac{x^3}{24}\right) \Big|_1^2 dy \\ &= \int_0^{\frac{1}{2}} \left(2y^2 + \frac{1}{3}\right) - \left(\frac{y^2}{2} + \frac{1}{24}\right) dy \\ &= \int_0^{\frac{1}{2}} \left(2y^2 + \frac{1}{3} - \frac{y^2}{2} - \frac{1}{24}\right) dy \\ &= \int_0^{\frac{1}{2}} \left(\frac{3y^2}{2} + \frac{7}{24}\right) dy = \frac{5}{24} \end{aligned}$$

$$\Rightarrow P\left(X > 1/Y < \frac{1}{2}\right) = \frac{5}{24}$$

$$P(Y < \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx dy$$

$$\begin{aligned}
 &= \int_0^{\frac{1}{2}} \left(\frac{x^2 y^2}{2} + \frac{x^3}{24} \right) dy \\
 &= \int_0^{\frac{1}{2}} \left(2y^2 + \frac{1}{3} \right) dy \\
 &= \left(\frac{2y^3}{3} + \frac{y}{3} \right) \Big|_0^{\frac{1}{2}} = \frac{1}{4}
 \end{aligned}$$

$$\therefore P\left(Y < \frac{1}{2}\right) = \frac{1}{4}$$

$$\begin{aligned}
 P(X > 1) &= \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy \\
 &= \int_0^1 \left(\frac{x^2 y^2}{2} + \frac{x^3}{24} \right) dy \\
 &= \int_0^1 \left(\frac{3y^2}{2} + \frac{7}{24} \right) dy \\
 &= \left[\frac{3y^3}{2 \cdot 3} + \frac{7y}{24} \right]_0^1 = \frac{19}{24}
 \end{aligned}$$

$$(i) P\left(X > 1 / Y < \frac{1}{2}\right) = \frac{P(X > 1, Y < \frac{1}{2})}{P(Y < \frac{1}{2})}$$

$$= \frac{5}{24} \times 4 = \frac{5}{6}$$

$$(ii) P\left(Y < \frac{1}{2} / X > 1\right) = \frac{P(Y < \frac{1}{2}, X > 1)}{P(X > 1)} = \frac{5}{24} \times \frac{24}{19} = \frac{5}{19}$$

$$(iii) P(X < Y) = \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8} \right) dx dy$$

$$= \int_0^1 \left(\frac{x^2 y^2}{2} + \frac{x^3}{24} \right) dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24} \right) dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{96} \right]_0^1 = \frac{53}{480}$$

$$(iv) P(X + Y \leq 1) = \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8} \right) dx dy$$

$$= \int_0^1 \left(\frac{x^2 y^2}{2} + \frac{x^3}{24} \right) dy$$

$$= \int_0^1 \left(\frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right) dy$$

$$= \int_0^1 \left(\frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} (y^2 + y^4 - 2y^3) + \frac{1}{24} (1-y)^3 \right) dy$$

$$= \frac{1}{2} \left(\frac{y^3}{3} + \frac{y^5}{5} - \frac{2y^4}{4} \right) + \frac{1}{24} \left(\frac{(1-y)^4}{-4} \right) \Big|_0^1 = \frac{13}{480}$$

4. The joint density function of X and Y is $f(x, y) = \begin{cases} e^{-(x+y)} & 0 \leq x, y \leq \infty \\ 0 & \text{otherwise} \end{cases}$.

Are X and Y independent. Find (i) $P(X < 1)$ (ii) $P(X + Y < 1)$

Solution:

$$\text{Given } f(x, y) = \begin{cases} e^{-(x+y)} & 0 \leq x, y \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

The Marginal pdf of X is $f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x, y) dy$

$$= \int_0^{\infty} e^{-(x+y)} dy$$

$$= e^{-x} \int_0^{\infty} e^{-y} dy$$

$$= e^{-x} [-e^{-y}]_0^{\infty}$$

$$= e^{-x} [0 + 1] = e^{-x}$$

Marginal pdf of 'x' is $f(x) = e^{-x}$

The Marginal pdf of Y is $f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x, y) dx$

$$= \int_0^{\infty} e^{-(x+y)} dx$$

$$= e^{-y} \int_0^{\infty} e^{-x} dx$$

$$= e^{-y} [-e^{-x}]_0^{\infty}$$

$$= e^{-y} [0 + 1] = e^{-y}$$

Marginal pdf of 'Y' is $f(y) = e^{-y}$

Now $f(x).f(y) = e^{-x}.e^{-y} = e^{-(x+y)}$

But $f(x, y) = e^{-(x+y)}$

$$\therefore f(x, y) = f(x).f(y)$$

Hence X and Y are independent.

$$(i) P(X < 1) = \int_0^1 \int_0^{\infty} e^{-(x+y)} dy dx$$

$$= \int_0^1 e^{-x} \left[\frac{e^{-y}}{-1} \right]_0^{\infty} dx$$

$$= \int_0^1 e^{-x} \left[\frac{0-1}{-1} \right] dx$$

$$= \int_0^1 e^{-x} dx$$

$$= [-e^{-x}]_0^1$$

$$= -e^{-1} + e^0 = 1 - e^{-1}$$

$$(ii) P(X + Y < 1) = \int_0^1 \int_0^{1-x} e^{-(x+y)} dy dx$$

$$= \int_0^1 e^{-x} \left[\frac{e^{-y}}{-1} \right]_0^{1-x} dx$$

$$= - \int_0^1 e^{-x} [e^{x-1} - e^0] dx$$

$$= - \int_0^1 e^{-x} [e^{x-1} - e^0] dx$$

$$= - \int_0^1 [e^{-x+x-1} - e^{-x}] dx$$

$$= -[e^{-1}x + e^{-x}]_0^1$$

$$= -[e^{-1} + e^{-1} - 0 - e^0]$$

$$= -[2e^{-1} - 1] = 1 - 2e^{-1}$$

5. If the joint pdf of X and Y is given by $f(x, y) =$

$$\begin{cases} \frac{1}{8}(6 - x - y) & 0 < x < 2, 2 < y < 4 \\ 0 & \text{otherwise} \end{cases} \quad \text{Find the value of (i) } P(X < 1 \cap Y < 3)$$

3) (ii) $P(X < 1 / Y < 3)$

Solution:

We know that (i) $P(X < 1 \cap Y < 3) = \int_0^1 \int_2^3 f(x, y) dy dx$

$$= \int_0^1 \int_2^3 \frac{1}{8} (6 - x - y) dy dx$$

$$= \frac{1}{8} \int_0^1 \left[6y - xy - \frac{y^2}{2} \right]_2^3 dx$$

$$= \frac{1}{8} \int_0^1 \left\{ \left(18 - 3x - \frac{9}{2} \right) - \left(12 - 2x - \frac{4}{2} \right) \right\} dx$$

$$= \frac{1}{8} \int_0^1 \left(8 - x - \frac{9}{2} \right) dx$$

$$= \frac{1}{8} \int_0^1 \left(\frac{7-2x}{2} \right) dx$$

$$= \frac{1}{16} [7x - x^2]_0^1 = \frac{6}{16} = \frac{3}{8}$$

$$\therefore P(X < 1 \cap Y < 3) = \frac{3}{8} \quad \dots (1)$$

$$(ii) P(X < 1 / Y < 3) = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)} \quad \dots (2)$$

$$= \int_0^1 \int_2^3 \frac{1}{8} (6 - x - y) dy dx$$

$$\begin{aligned}
 &= \frac{1}{8} \int_0^2 [6y - xy - \frac{y^2}{2}]_2^3 dx \\
 &= \frac{1}{8} \int_0^2 \{ (18 - 3x - \frac{9}{2}) - (12 - 2x - 2) \} dx \\
 &= \frac{1}{8} \int_0^2 (8 - x - \frac{9}{2}) dx \\
 &= \frac{1}{8} \int_0^2 \{ (\frac{16-9}{2}) - x \} dx \\
 &= \frac{1}{8} [\int_0^2 (\frac{7}{2} - x) dx] = \frac{1}{8} [\frac{7}{2}x - \frac{x^2}{2}]_0^2 = \frac{5}{8}
 \end{aligned}$$

$$\therefore P(Y < 3) = \frac{5}{8} \dots (3)$$

Substituting (1) and (3) in (2) we get $P(X < 1/Y < 3) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$

6. If the joint distribution function of X and Y is given by $f(x, y) =$

$$\begin{cases} (1 - e^{-x})(1 - e^{-y}) & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{(i) Find the marginal densities of X and Y.}$$

and Y.

(ii) Are X and Y independent? (iii) $P(1 < X < 3, 1 < Y < 2)$

Solution:

$$\begin{aligned}
 \text{Given } f(x, y) &= (1 - e^{-x})(1 - e^{-y}) \\
 &= 1 - e^{-x} - e^{-y} + e^{-(x+y)}
 \end{aligned}$$

$$\begin{aligned} \text{The joint pdf is given by } f(x, y) &= \frac{\partial^2 F(x, y)}{\partial x \partial y} \\ &= \frac{\partial^2}{\partial x \partial y} [1 - e^{-x} - e^{-y} + e^{-(x+y)}] \\ &= \frac{\partial}{\partial x} [e^{-y} + e^{-(x+y)}] = 0 + e^{-(x+y)} \end{aligned}$$

$$\therefore f(x, y) = \begin{cases} (1 - e^{-x})(1 - e^{-y}) & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x, y) = e^{-(x+y)}$$

The marginal pdf of X is $f_{X(x)} = f(x) = \int_{-\infty}^{\infty} f(x, y) dy$

$$\begin{aligned} &= \int_0^{\infty} e^{-(x+y)} dy \\ &= [-e^{-(x+y)}]_0^{\infty} \\ &= -e^{-\infty} + e^{-x} = e^{-x} \quad \dots (1) \end{aligned}$$

The marginal pdf of Y is $f_{Y(y)} = f(y) = \int_{-\infty}^{\infty} f(x, y) dx$

$$\begin{aligned} &= \int_0^{\infty} e^{-(x+y)} dx \\ &= [-e^{-(x+y)}]_0^{\infty} \\ &= -e^{-\infty} + e^{-y} = e^{-y} \quad \dots (2) \end{aligned}$$

(ii) From (1) and (2) we get

$$f(x) \cdot f(y) = e^{-x} e^{-y} = e^{-(x+y)} = f(x, y)$$

Hence X and Y are independent.

$$(iii) P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3) \cdot P(1 < Y < 2)$$

(Since X and Y are independent)

$$= \int_1^3 e^{-x} dx \cdot \int_1^2 e^{-y} dy$$

$$= (-e^{-x})_1^3 (-e^{-y})_1^2$$

$$= (-e^{-3} + e^{-1})(-e^{-2} + e^{-1})$$

$$= e^{-5} - e^{-4} - e^{-3} + e^{-2}$$

