Random Process:

Consider a random experiment with a sample space S. If a time function X(t, s) is assigned to each outcome $s \in S$ and where $t \in T$, then the family of all such functions, denoted by $\{X(t, s)\}$, where $s \in S$, $t \in T$ is called a random process. In other words, a random process is a collection of random variables together with time.

Note: A random process is also called stochastic process.

1 Classification of Random Process:

Classify a random process according to the characteristic of T and the state space S. We shall consider only 4 cases based on T and S.

i) Continuous random process
ii) Continuous random sequence
iii) Discrete random process
iv) Discrete random sequence

Continuous random Process:

If both S and T are continuous, then the random process is called continuous Random process.

Continuous Random Sequence:

If S is continuous and T is discrete, then the random process is called continuous random sequence.

Discrete Random Process:

If S is discrete and T is continuous, then the random process is called discrete random process.

Discrete Random Sequence:

If both S and T are discrete, then the random process is called discrete random process.

Deterministic Random Process:

A random process is called a deterministic random process if all the future values are predicted from past observation.

Non Deterministic Random Process:

A random process is called a non - deterministic random process if the future values of any sample function cannot be predicted from the past observation.

Wide Sense Stationary Process (WSS);

A process $\{(t)\}$ is said to be Wide Sense Stationary Process if(i)$Mean$

$$= E[X(t)] = constant$$

(ii) $Auto\ correlation\ R_X(r) = E[X(t)X(t+r)]$ depends on r

Note:

A WSS process is also called as Weak Sense Stationary Process.

A SSS process is also called a strongly stationary process.

For stationary process mean and variance are constants.

A random process, which is not stationary in any sense, is called evolutionary.

Formulae:

Wide Sense Stationary (WSS):

(i)$Mean = E[X(t)] = constant$

(ii) $Auto\ correlation\ R_X(r) = E[X(t)X(t+r)]$ depends on r

Stationary Process:

(i)$E[X(t)] = constant$

(ii)$Var[X(t)] = constant$

Strict Sense Stationary (SSS):

$E[X^n(t)]\ is\ a\ constant\ for\ every\ n$

Joint Wide Sense Stationary (JWSS):

(i)$E[X(t)] = constant$

(ii)$E[Y(t)] = constant$

(iii)$R_{XX}(t, t+r) = E[X(t)Y(t+r)]\ depends\ on\ r$

Mean Ergodic:

Time average, $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt$

$[X(t)] = \lim_{T\to\infty} \overline{X}_T$

Correlation Ergodic:

$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\, X(t+r)dt$

$R_X(t, t+r) = \lim_{T\to\infty} \overline{X}_T$

If $(t) = X(t+a) - X(t-a)$, prove that $R_{YY}(r) = \langle 2R_{XX}(r) - R_{XX}(r+2a) - R_{XX}(r-2a)$

Solution:

Given $(t) = X(t+a) - X(t-a)$

$R_{YY}(t)= E[Y(t_1)Y(t_2)]$

$= [(X(t_1+a) - X(t_1-a)(X(t_2+a) - X(t_2-a))]$

$= E[(X(t_1+a)(X(t_2+a) - X(t_1+a)X(t_2-a) - X(t_1-a)(X(t_2 \quad +a) + (t_1-a)X(t_2-a))]$

$= E[X(t_1+a)(X(t_2+a)] - E[X(t_1+a)X(t_2-a)] - E[X(t_1$

$-a)(X(t_2+a)] + E[X(t_1-a)X(t_2-a)]$

$=R_X(t_1+a, t_2+a) - R_{XX}(t_1+a, t_2-a) - R_{XX}(t_1-a, t_2+a) + R_X(t_1-a, t_2-a)$

$=R_X(t_1+a-t_2-a) - R_{XX}(t_1+a-t_2+a) - R_{XX}(t_1-a-t_2-a)$

$+R_X(t_1-a-t_2+a)$

$=R_X(t_1-t_2) - R_{XX}(t_1-t_2+2a) - R_{XX}(t_1-t_2-2a) + R_{XX}(t_1-t_2)$

$=R_X(r) - R_{XX}(r+2a) - R_{XX}(r-2a) + R_{XX}(r)$

$R_{YY}(r) = 2R_X(r) - R_{XX}(r+2a) - R_{XX}(r-2a)$

The following formulas are very useful to solve problems under stationary process.

> If X is a RV with mean zero, then $\text{Var}(X) = E(X^2)$

> $1 + 2x + 3x^2 + \cdots = (1-x)^{-2}$

> $1 + 4x + 9x^2 + \cdots = (1+x)(1-x)^{-3}$

> If A and B are RV's and λ is a constant, then

$[A\cos\lambda t + B\sin\lambda t] = E(A)\cos\lambda t + E(B)\sin\lambda t$

$> \therefore (\cos\lambda r) = \cos\lambda r$, since λ and r are constants.

STATIONARY PROCESS

Problems under Stationary process:

For a stationary process

(1) $E[X(t)]$ is a constant

(2) $\text{Var}[X(t)]$ is a constant

1. The process $\{(t)\}$ whose probability distribution unde certain conditions is given by $P[X(t) = n] = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{(n+1)}}; n = 1, 2, 3, \ldots . \\ \frac{at}{1+at}; n = 0 \end{cases}$

Show that it is not a stationary process (Evolutionary).

Solution:

n	0	1	2	3	...
$p_n(t)$	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	...

For a stationary process,

(1) $E[X(t)]$ is a a constant

(2) Var$[X(t)]$ is a constant

$$E[X(t)] = \sum_{n=0} np_n(t) = 0 + \frac{1}{(1+at)^2} + (2)\frac{at}{(1+at)^3} + (3)\frac{(at)^2}{(1+at)^4}..$$

$$= \frac{1}{(1+at)^2}[1 + 2\frac{t}{1+at} + 3\frac{(at)^2}{(1+at)^2} + \cdots \ldots \ldots]$$

$$= \frac{1}{(1+at)^2}[1 - \frac{at}{(1+at)}]^{-2}$$

$$= \frac{1}{(1+at)^2}[\frac{1+at-at}{(1+at)}]^{-2}$$

$$= \frac{1}{(1+at)^2}[\frac{1}{(1+at)}]^{-2}$$

$$= \frac{1}{(1+at)^2}(1+at)^2 = 1$$

$[X\ (t)] = 1$ which is a constant

$$E[X^2(t)] = \sum_{n=1} n^2 p_n(t) = 0 + \frac{1}{(1+at)^2} + (4)\frac{at}{(1+at)^3} + (9)\frac{(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{(1+at)^2}[1 + 4\frac{t}{1+at} + 9\frac{(at)^2}{(1+at)^2} + \cdots]$$

$$= \frac{1}{(1+at)^2}(1 + \frac{t}{1+at})[1 - \frac{t}{1+at}]$$

$$= \frac{1}{(1+at)^2}(\frac{1+2at}{1+at})(1+at)^3$$

$E[X^2(t)] = 1 + 2at$, which is not a constant

Var [X(t)] $= E[X^2(t)] - [E[X(t)]]^2 = 1 + 2at - 1$

$= 2at$ which is not a constant.

$\therefore \{(t)\}$ is not a stationary process.

2. Consider a random process A_1 and A_2 are independent random variables with $E(A_i) = a_i$ and Var$(A_i) = \sigma^2{}_i$ for $i = 1, 2$ Prove that the process $\{X(t)\}$ is evolutionary.

Solution:

Given $(t) = A_1 + A_1 t$ where A_1 and A_2 are independent random variables with $E(A_i) = a_i$ and Var$(A_i) = \sigma^2{}_i$ for $i = 1,2$

For a stationary process

(1) $E[X(t)]$ is a constant

(2) Var$[X(t)]$ is a constant

$[X(t)] = E[A_1 + A_1 t]$

$= [A_1] + tE[A_2]$

$= a_1 + ta_2$

Which is not a constant.

Thus, the process {X(t)} is evolutionary.

3. Let $(t) = B\sin mt$, where B is a random variable with mean and variance 1 and m is a constant. Check whether $\{X(t)\}$ is a stationary or not

Solution:

Given $(t) = B\sin \omega t$, where

B is a random variable with Mean=0 and Variance =1

Mean of $B = 0 \Rightarrow (B) = 0$ .... (i)

Variance of $B = 1 \Rightarrow (B^2) = 1$ .... (ii)

For a stationary process,

(1) $E[X(t)]$ is a constant

(2) Var$[X(t)]$ is a constant

(1) $[X(t)] \; = E[B\sin \omega t]$

$= [B]\sin \omega t$

$= 0$ From (i)

$\therefore [X(t)]$ is a constant

(2) $[X^2(t)] = E[B^2\sin^2 \omega t]$

$= (B^2)\sin^2 \omega t$

$= \sin^2 \omega t$ which is not a constant From (ii)

$\text{Var}[X(t)] = E[X^2(t)] - [E[X(t)]^2] = \sin^2 \omega t$, which is not a constant.

Since the condition (2) for Stationary Process is not satisfied,

Hence $\{(t)\}$ is not a Stationary Process.

4. Consider the random process $X(t) = \cos(t + \varphi)$ where φ is a random variable with density function $f(\varphi) = \frac{1}{\pi}$, where $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. Check whether or not the process is stationary.

Solution:

$(t) = \cos(t + \varphi)$ where φ is a random variable with

$(\varphi) = \frac{1}{\pi}$, where $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$

For a stationary process,

(1) $E[X(t)]$ is a constant

(2) $\text{Var}[X(t)]$ is a constant

$\text{E}\,[X(t)] = E[\cos(t + \varphi)]$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) f(\varphi) d\varphi$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) \frac{1}{\pi} d\varphi$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t + \varphi) d\varphi$$

$$= \frac{1}{\pi}[\sin(t+\varphi)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}[\sin(t+\frac{\pi}{2}) - \sin(t-\frac{\pi}{2})]$$

$$= \frac{1}{\pi}[\sin(\frac{\pi}{2}+t) + \sin(\frac{\pi}{2}-t)]$$

$$\because \sin(\frac{\pi}{2}-\theta) = \sin(\frac{\pi}{2}+\theta) = \cos\theta$$

$$= \frac{1}{\pi}[\cos t + \cos t]$$

$$= \frac{1}{\pi}2\cos t$$

$E[X(t)] = \frac{1}{\pi}2\cos t$, which depends on t.

Since the condition (1) for Stationary Process is not satisfied,

$\{(t)\}$ is not a Stationary Process.

5. Let $(t) = \cos(mt+\theta)$, where θ is a random variable uniformly distributed over $(0, 2\pi)$. Prove that $\{(t)\}$ is a stationaryprocess of first order.

Solution:

Given: $(t) = \cos(\omega t+\theta)$, where θ is random variable uniformly distributed over $(0,2\pi)$.

$$\therefore f_\theta(\theta) = \frac{1}{2\pi}; 0 < \theta < 2\pi$$

To prove $\{(t)\}$ is a first order stationary process.

we have to prove $f_X(x;t)$ is independent of time.

To find $f(x;t)$:

We have $x = \cos(\omega t + \theta)$

$\Rightarrow \omega t + \theta = \pm\cos^{-1}[x]$

To find $f(x;t)$,

Take $x = (t)$

$\Rightarrow \theta = -\omega t \pm \cos^{-1}[x] \because \cos[\pm(\omega t + \theta)] = \cos(\omega t + \theta$

Let $\theta_1 = -\omega t - \cos^{-1} x$ and $\theta_2 = -\omega t + \cos^{-1} x$

$$\frac{d\theta_1}{dx} = 0 - \frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\theta_1}{dx} = 0 + \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}}$$

The first order density of $\{(t)\}$ is given by

$$f_x(x,t) = \left|\frac{d\theta_1}{dx}\right| f_\theta(\theta_1) + \left|\frac{d\theta_2}{dx}\right| f_\theta(\theta_2)$$

$$= \left|\frac{1}{\sqrt{1-x^2}}\right| \frac{1}{2\pi} + \left|\frac{-1}{\sqrt{1-x^2}}\right| \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{2}{2\pi}\frac{1}{\sqrt{1-x^2}}$$

$$f_X(x,t) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$$

We have $x = (t) = \cos(\omega t + \theta)$.

Since the value of $\cos(\omega t + \theta)$ lies between -1 and $+1$, we have $-1 \leq x \leq 1$.

$$\therefore f_X(x, t) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, -1 \leq x \leq 1 \ldots \ldots (1)$$

which is independent of time.

Hence, $\{(t)\}$ is a stationary process of first order.