M circles

Constant-magnitude loci (M-circles)

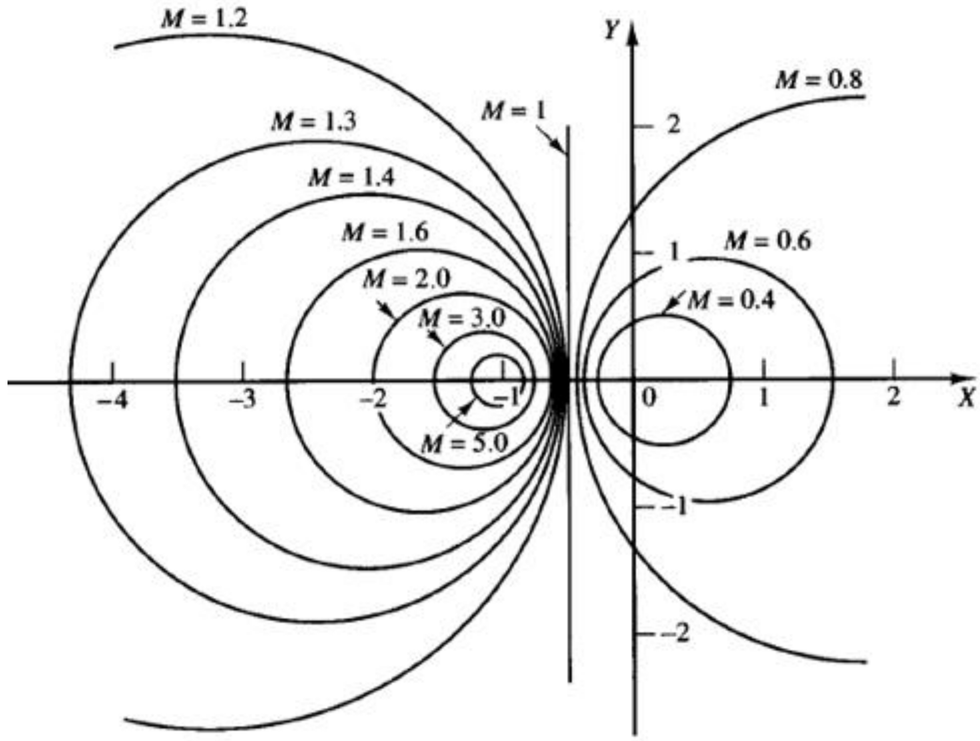

Fig: 1 A family of constant M circles

To obtain the constant-magnitude loci, let us first note that G(jω) is a complex quantity and can be written as follows:

$$G(j\omega) = X + jY$$

where X and Y are real quantities. Then M is given by

$$M = \frac{|X + jY|}{|1 + X + jY|}$$

and M^2 is

$$M^2 = \frac{X^2 + Y^2}{(1 + X)^2 + Y^2}$$

Hence

$$X^2(1 - M^2) - 2M^2X - M^2 + (1 - M^2)Y^2 = 0$$

If M = 1, then from the above equation we obtain X = -1/2

This is the equation of a straight line parallel to the Y-axis and passing through the point (-t, 0).

If M ≠ 1, the above equation can be written

$$X^2 + \frac{2M^2}{M^2 - 1}X + \frac{M^2}{M^2 - 1} + Y^2 = 0$$

If the term $M^2/(M^2 - 1)^2$ is added to both sides of this last equation, we obtain

$$\left(X + \frac{M^2}{M^2 - 1}\right)^2 + Y^2 = \frac{M^2}{(M^2 - 1)^2}$$

The above equation is the equation of a circle with center at $X = -M^2/(M^2 - 1)$, Y =0 and with radius $|M/(M^2 - 1)|$.

The constant M loci on the G(s) plane are thus a family of circles.

The center and radius of the circle for a given value of M can be easily calculated.

For example, for M = 1.3, the center is at (- 2.45,0) and the radius is 1.88.

A family of constant M circles is shown in Figure 1.

It is seen that as M becomes larger compared with 1, the M circles become smaller and converge to the -1 j0 point.

For M > 1, the centers of the M circles lie to the left of the -1 j0 point.

Similarly, as M becomes smaller compared with 1, the M circle becomes smaller and converges to the origin.

For 0 < M < 1, the centers of the M circles lie to the right of the origin.

M = 1 corresponds to the locus of points equidistant from the origin and from the -1 j0 point.

As stated earlier, it is a straight line passing through the point (-t, 0) and parallel to the imaginary axis. (The constant M circles corresponding to M > 1 lie to the left of the M = 1 line and those corresponding to 0< M < 1 lie to the right of the M = 1 line.)

The M circles are symmetrical with respect to the straight line corresponding to M = 1 and with respect to the real axis.

N circles

Constant phase-angle loci (N-circles):

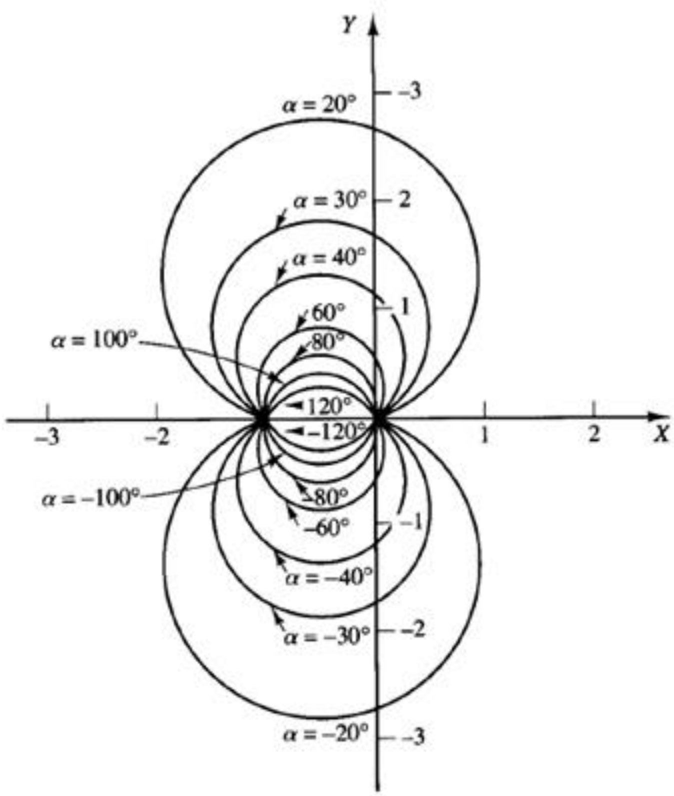

Fig 1: A family of constant N circles

We shall obtain the phase angle α in terms of X and Y. Since

$$\angle e^{j\alpha} = \angle \frac{X + jY}{1 + X + jY}$$

the phase angle α is

$$\alpha = \tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1 + X}\right)$$

If we define

$$\tan \alpha = N$$

then

$$N = \tan\left[\tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1 + X}\right)\right]$$

Since

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

we obtain

$$N = \frac{\frac{Y}{X} - \frac{Y}{1 + X}}{1 + \frac{Y}{X}\left(\frac{Y}{1 + X}\right)} = \frac{Y}{X^2 + X + Y^2}$$

or

$$X^2 + X + Y^2 - \frac{1}{N} Y = 0$$

The addition of (¼) + $1/(2N)^2$ to both sides of this last equation yields

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2$$

This is an equation of a circle with center at X = -1/2 , y = 1/(2N) and with radius $\sqrt{(1/4) + 1/(2N)^2}$.

For example, if α = 30°, then N = tan α = 0.577, and the center and the radius of the circle corresponding to α = 30° are found to be (-0.5,0.866) and unity, respectively.

Since the above equation is satisfied when X = Y = 0 and X = -1, Y = 0 regardless of the value of N, each circle passes through the origin and the -1 j0 point.

The constant a loci can be drawn easily once the value of N is given.

A family of constant N circles is shown in Figure 1 with α as a parameter.

It should be noted that the constant N locus for a given value of a is actually not the entire circle but only an arc.

In other words, α = 30° and α = -150° arcs are parts of the same circle.

This is so because the tangent of an angle remains the same if ± 180° (or multiplies thereof) is added to the angle.