BOUNDARY LAYER ON VERTICAL PLATE

Analytical Solution-Flow over a Heated Vertical Plate in Air

Let us consider a heated vertical plate in air, shown in Fig. 2.5. The plate is maintained at uniform temperature T_w .The coordinates are chosen in such a way that x - is in the stream wise direction and y - is in the transverse direction. There will be a thin layer of fluid adjacent to the hot surface of the vertical plate within

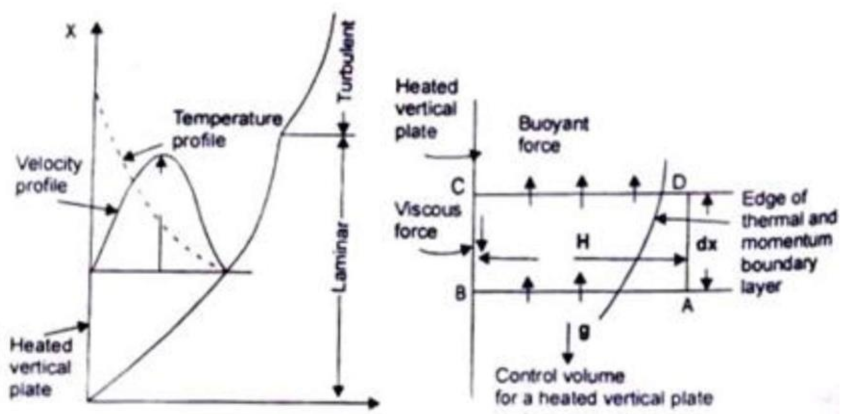

Fig. 2.5 Boundary layer on a heated vertical plate

Which the variations in velocity and temperature would remain confined. The relative thickness of the momentum and the thermal boundary layer strongly depends upon the Prandtl number. Since in natural convection heat transfer, the motion of the fluid particles is caused by the temperature difference between the temperatures of the wall and the ambient fluid, the thickness of the two boundary layers are expected to be equal. When the temperature of the vertical plate is less than the fluid temperature, the boundary layer will form from top to bottom but the mathematical analysis will remain the same.

The boundary layer will remain laminar upto a certain length of the plate ($Gr < 10^8$) and beyond which it will become turbulent ($Gr > 10^9$). In order to obtain the analytical solution, the integral approach, suggested by von-Karman is preferred.

We choose a control volume ABCD, having a height H, length dx and unit thickness normal to the plane of paper, as shown in Fig. 25. We have:

(b) Conservation of Mass:

Mass of fluid entering through face AB = $\dot{m}_{AB} = \int_0^H \rho u dy$

Mass of fluid leaving face CD = $\dot{m}_{CD} = \int_0^H \rho u dy + \frac{d}{dx}\left[\int_0^H \rho u dy\right]dx$

$\therefore$ Mass of fluid entering the face DA = $\frac{d}{dx}\left[\int_0^H \rho u dy\right]dx$

(ii) Conservation of Momentum:

Momentum entering face AB = $\int_0^H \rho u^2 dy$

Momentum leaving face CD = $\int_0^H \rho u^2 dy + \frac{d}{dx}\left[\int_0^H \rho u^2 dy\right]dx$

$\therefore$ Net efflux of momentum in the + x-direction = $\frac{d}{dx}\left[\int_0^H \rho u^2 dy\right]dx$

The external forces acting on the control volume are:

(a) Viscous force = $\mu \left.\frac{du}{dy}\right|_{y=0} dx$ acting in the –ve x-direction

(b) Buoyant force approximated as $\left[\int_0^H \rho g \beta (T - T_\infty) dy\right]dx$

From Newton's law, the equation of motion can be written as:

$$\frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right] = -\mu \left.\frac{du}{dy}\right|_{y=0} + \int_0^\delta \rho g \beta (T - T_\infty) dy \qquad (2.2)$$

because the value of the integrand between δ and H would be zero.

(iii) Conservation of Energy:

$\dot{Q}_{AB}$, convection + $\dot{Q}_{AD}$, convection + $\dot{Q}_{BC}$, conduction = $\dot{Q}_{CD}$ convection

or, $\int_0^H \rho u C T dy + C T_\infty \left[\frac{d}{dx}\int_0^H \rho u dy\right]dx - k \left.\frac{dT}{dy}\right|_{y=0} dx$

$= \int_0^H \rho u C T dy + \frac{d}{dx}\left[\int_0^H \rho u T C dy\right]dx$

or $\frac{d}{dx}\int_0^{\delta}\rho u\left(T_{\infty}-T\right)dy\ \frac{k}{\rho C}\frac{dT}{dy}\bigg|_{y=0}=\alpha\frac{dT}{dy}\bigg|_{y=0}$ (2.3)

The boundary conditions are:

or,

(2.3)

Velocity profile	Temperature profile
$u = 0$ at $y = 0$	$T = T_w$ at $y = 0$
$u = 0$ at $y = \delta$	$T = T_{\infty}$ at $y = \delta_1 \equiv \delta$
$du/dy = 0$ at $y = \delta$	$dT/dy \equiv 0$ at $y = \delta_1 \equiv \delta$

Since the equations (2.2) and (2.3) are coupled equations, it is essential that the functional form of both the velocity and temperature distribution are known in order to arrive at a solution.

The functional relationships for velocity and temperature profiles which satisfy the above boundary conditions are assumed of the form:

$$\frac{u}{u_*}=\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2 \qquad (2.4)$$

Where u_* is a fictitious velocity which is a function of x; and

$$\frac{\left(T-T_{\infty}\right)}{\left(T_w-T_{\infty}\right)}=\left(1-\frac{y}{\delta}\right)^2 \qquad (2.5)$$

After the Eqs. (5.4) and (5.5) are inserted in Eqs. (5.2) and (5.3) and the operations are performed (details of the solution are given in Chapman, A.J. Heat Transfer, Macmillan Company, New York), we get the expression for boundary layer thickness as:

$$\delta/x = 3.93\mathrm{Pr}^{-0.5}\left(0.952+\mathrm{Pr}\right)^{0.25}\mathrm{Gr}_x^{-0.25}$$

Where Gr, is the local Grashof number = $g\beta x^3\left(T_w-T_{\infty}\right)/\nu^2$

The heat transfer coefficient can be evaluated from:

$$\dot{q}_w = -k \left.\frac{dT}{dy}\right|_{y=0} = h(T_w - T_\infty)$$

Using Eq. (5.5) which gives the temperature distribution, we have

$h = 2k/\delta$ or, $hx/k = Nu_x = 2x/\delta$

The non-dimensional equation for the heat transfer coefficient is

$$Nu_x = 0.508\ Pr^{0.5}\ (0.952 + Pr)^{-0.25}\ Gr_x^{0.25} \qquad (2.7)$$

The average heat transfer coefficient, $\bar{h} = \frac{1}{L}\int_0^L h_x\, dx = 4/3 h_{x=L}$

$$Nu_L = 0.677\ Pr^{0.5}\ (0.952 + Pr)^{-0.25}\ Gr^{0.25} \qquad (2.8)$$

Limitations of Analytical Solution: Except for the analytical solution for flow over a flat plate, experimental measurements are required to evaluate the heat transfer coefficient. Since in free convection systems, the velocity at the surface of the wall and at the edge of the boundary layer is zero and its magnitude within the boundary layer is so small. It is very difficult to measure them. Therefore, velocity measurements require hydrogen-bubble technique or sensitive hot wire anemometers. The temperature field measurement is obtained by interferometer.

Expression for 'h' for a Heated Vertical Cylinder in Air

The characteristic length used in evaluating the Nusselt number and Grashof number for vertical surfaces is the height of the surface. If the boundary layer thickness is not to large compared with the diameter of the cylinder, the convective heat transfer coefficient can be evaluated by the equation used for vertical plane surfaces. That is, when $D/L \geq 25/(Gr_L)^{0.25}$

Example 2.1 A large vertical flat plate 3 m high and 2 m wide is maintained at 75°C and is exposed to atmosphere at 25°C. Calculate the rate of heat transfer.

Solution: The physical properties of air are evaluated at the mean temperature. i.e. T = (75 + 25)/2 = 50°C

From the data book, the values are:

$\rho = 1.088$ kg/m^3; $C_p = 1.00$ kJ/kg.K;

$\mu = 1.96 \times 10^{-5}$ Pa-s k = 0.028 W/mK.

$Pr = \mu C_p/k = 1.96 \times 10^{-5} \times 1.0 \times 10^3 /0.028 = 0.7$

$$\beta = \frac{1}{T} = \frac{1}{323}$$

$$Gr = \rho^2 g\beta(\Delta T)L^3/\mu^2$$

$$= \frac{(1.088)^2 \times 9.81 \times 1 \times (3)^3 \times 50}{323 \times (1.96 \times 10^{-5})^2}$$

$= 12.62 \times 10^{10}$

$Gr.Pr = 8.834 \times 10^{10}$

Since Gr.Pr lies between 10^9 and 10^{13}

We have from Table 2.1

$$Nu = \frac{hL}{k} = 0.1(Gr.Pr)^{1/3} = 441.64$$

$\therefore h = 441.64 \times 0.028/3 = 4.122$ W/m^{2}K

$$\dot{Q} = hA(\Delta T) = 4.122 \times 6 \times 50 = 1236.6W$$

We can also compute the boundary layer thickness at x = 3m

$$\delta = \frac{2x}{Nu_x} = \frac{2 \times 3}{441.64} = 1.4 \text{ cm}$$

Example 2.2 A vertical flat plate at 90°C. 0.6 m long and 0.3 m wide, rests in air at 30°C. Estimate the rate of heat transfer from the plate. If the plate is immersed in water at 30°C. Calculate the rate of heat transfer

Solution: (a) *Plate in Air*: Properties of air at mean temperature 60°C

Pr = 0.7, k = 0.02864 W/ mK, $v = 19.036 \times 10^{-6}$ m^2/s

$Gr = 9.81 \times (90 - 30)(0.6)^3/ [333 (19.036 \times 10^{-6})^2]$

$= 1.054 \times 10^9$; $Gr \times Pr$ $1.054 \times 10^9 \times 0.7 = 7.37 \times 10^8 < 10^9$

From Table 5.1: for $Gr \times Pr < 10^9$, $Nu = 0.59\,(Gr.\,Pr)^{1/4}$

$\therefore\ h = 0.02864 \times 0.59\,(7.37 \times 10^8)^{1/4}/0.6 = 4.64\ W/m^2K$

The boundary layer thickness, $\delta = 2\,k/h = 2 \times 0.02864/4.64 = 1.23$ cm

and $\dot{Q} = hA\,(\Delta T) = 4.64 \times (2 \times 0.6 \times 0.3) \times 60 = 100$ W.

Using Eq (2.8). $Nu = 0.677\,(0.7)^{0.5}\,(0.952 + 0.7)^{0.25}\,(1.054 \times 10^9)^{0.25}$,

Which gives $h = 4.297\ W/m^2K$ and heat transfer rate, $\dot{Q}$ 92.81 W

Churchill and Chu have demonstrated that the following relations fit very well with experimental data for all Prandtl numbers.

For $Ra_L < 10^9$, $Nu = 0.68 + (0.67\,Ra_L^{0.25})/\,[1 + (0.492/Pr)^{9/16}]^{4/9)}$ (5.9)

$Ra_L > 10^9$, $Nu = 0.825 + (0.387\,Ra_L^{1/6})/[1 + (0.492/Pr)^{9/16}]^{8/27}$ (5.10)

Using Eq (5.9): $Nu = 0.68 + [0.67(7.37 \times 10^8)^{0.25}]\,/\,[1 + (0.492/0.7)^{9/16}]^{4/9}$

$= 58.277$ and $h = 4.07\ W/m^2k$; $\dot{Q} = 87.9$ W

(b) Plate in Water: Properties of water from the Table

$Pr = 3.01$, $\rho^2\, g\, \beta\, C_p/\mu k = 6.48 \times 10^{10}$;

$Gr.Pr = \rho^2\, g\, \beta\, C_p\, L^3(\Delta T)/\,\mu k = 6.48 \times 10^{10} \times (0.6)^3 \times 60 = 8.4 \times 10^{11}$

Using Eq (5.10): $Nu = 0.825 + [0.387\,(8.4 \times 10^{11})^{1/6}]/\,[1 + (0.492/3.01)^{9/16})]^{8/27} = 89.48$ which gives $h = 97.533$ and $Q = 2.107$ kW.

2.9. Modified Grashof Number

When a surface is being heated by an external source like solar radiation incident on a wall, a surface heated by an electric heater or a wall near a furnace, there is a uniform heat flux distribution along the surface. The wall surface will not be an isothermal one. Extensive experiments have been performed by many research workers for free convection from vertical and inclined surfaces to water under constant heat flux conditions. Since the temperature difference (ΔT) is not known beforehand, the Grashof number is modified by multiplying it by

Nusselt number. That is,

$$Gr_x^* = Gr_x . Nu_x = (g\ \beta\ \Delta T /\ \nu^2) \times (hx/k) = g\ \beta\ x^4\ q/k\,\nu^2 \qquad (2.11)$$

Where q is the wall heat flux in Wm^2. ($q = h\ (\Delta T)$)

It has been observed that the boundary layer remains lam mar when the modified Rayleigh number, $Ra^* = Gr_x^*$ '. Pr is less than 3×10^{12} and fully turbulent flow appears for $Ra^* > 10^{14}$. The local heat transfer coefficient can be calculated from:

$$q \text{ constant and } 10^5 < Gr_x^* < 10^{11}: Nu_x = 0.60\ (Gr_x^* . Pr)^{0.2} \qquad (2.12)$$

$$q \text{ constant and } 2 \times 10^{13} < Gr_x^* < 10^{16} : Nu_x = 0.17\ (Gr_x^* . Pr)^{0.25} \qquad (2.13)$$

Although these results are based on experiments for water, they are applicable to air as well. The physical properties are to be evaluated at the local film temperature.

Example 2.3 Solar radiation of intensity 700W/m' is incident on a vertical wall, 3 m high and 3 m wide. Assuming that the wall does not transfer energy to the inside surface and all the incident energy is lost by free convection to the ambient air at 30oe, calculate the average temperature of the wall

Solution: Since the surface temperature of the wall is not known, we assume a value for $h = 7\ W/m^2 K$.

$\Delta T = \dot{q}\ / h = 700/7 = 100^oC$ and the film temperature = $(30 + 130)/2 = 80^oC$

The properties of air at 273 +80 = 353 are: $\beta = 1/353$, $Pr = 0.697$

$k = 0.03\ W/mK$, $\nu = 20.76 \times 10^{-6}\ m^2/s$.

Modified Grashof number, $Gr_x^* = 9.81. (1/353) \cdot (3)^4 \times 700/[0.03 \times (20.76 \times 10^{-6})^2] = 1.15 \times 10^{14}$

From Eq. (2.13), $h = (k/x)\ (0.17)\ (Gr_x^*\ Pr)^{0.25}$

$= (0.03/3)\ (0.17)\ (1.15 \times 10^{14} \times 0.697)^{1/4}$

$= 5.087\ W/m^2K$, a different value from the assumed value.

Second Trial: $\Delta T = \dot{q}\ / h = 700/5.087 = 137.66$ and film temperature

$= 98.8°C$

The properties of air at (273 + 98.8) °C are: $\beta = 1/372$, k = 0.0318 W/mK

Pr = 0.693, $v = 23.3 \times 10^{-6}\ m^2/s$

$Gr_x^* = 9.81.\ (1/372) \cdot (3)^4 \times 700/\ [0.318(23.3 \times 10^{-6})^2] = 8.6 \times 10^{13}$

Using Eq (2.13), $h = (k/x)\ (0.17)\ (\ Gr_x^*\ Pr)^{1/4} = 5.015\ W/m^2k$, an acceptable value. In turbulent heat transfer by convection, Eq. (5.13) tells us that the local heat transfer coefficient h_x does not vary with x and therefore, the average and local heat transfer coefficients are the same.