Poynting Vector and Power Flow in Electromagnetic Fields

Electromagnetic waves can transport energy as a result of their travelling or propagating characteristics. Starting from Maxwell's Equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Together with the vector identity

$$\nabla . \left(\vec{E} \times \vec{H} \right) = \vec{H} . \nabla \times \vec{E} - \vec{E} . \nabla \times \vec{H}$$

One can write

$$\nabla . \left(\vec{E} \times \vec{H} \right) = -\vec{H} . \frac{\partial \vec{B}}{\partial t} - \vec{E} . \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

$$or, \; \nabla . \left(\vec{E} \times \vec{H} \right) = -\vec{H} . \frac{\partial \vec{B}}{\partial t} - \vec{E} . \vec{J} - \vec{E} . \frac{\partial \vec{D}}{\partial t}$$

In simple medium where $\in, \mu$ and σ are constant,

$$\vec{H} . \frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} \mu H^2 \right)$$

$$\vec{E} . \frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} \mu E^2 \right) \quad \text{and} \quad \vec{E} . \vec{J} = \sigma E^2$$

$$\therefore \nabla . \left(\vec{E} \times \vec{H} \right) = -\frac{\partial}{\partial t} \left(\frac{1}{2} \in E^2 + \frac{1}{2} \mu H^2 \right) - \sigma E^2$$

Divergence theorem states,

$$\oint_S \left(\vec{E} \times \vec{H} \right) . d\vec{S} = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \in E^2 + \frac{1}{2} \mu H^2 \right) dV - \int_V \sigma E^2 dV$$

This equation is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

In the equation, the following term represents the rate of change of the stored energy in the electric and magnetic fields

$$\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) dV$$

On the other hand, the power dissipation within the volume appears in the following form

$$\int_V \sigma E^2 dV$$

Hence the total decrease in power within the volume under consideration:

$$\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = \oint_S \vec{P}.d\vec{S}$$

Here $\vec{P} = \vec{E}\times\vec{H}$ (W/mt^2) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface.

Poynting vector for the time harmonic case:

Using the $e^{j\omega t}$ convention, the instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. Considering the following phasor:

$$\vec{E}(z) = \hat{x}E_x(z) = \hat{x}E_0 e^{-j\beta z}$$

The instantaneous field becomes:

$$\vec{E}(z,t) = Re\{\vec{E}(z)e^{jwt}\} = \hat{x}E_0 cos(\omega t - \beta z)$$

when E_0 is real.

Let us consider two instantaneous quantities A and B such that

$$A = \mathrm{Re}\left(Ae^{j\omega t}\right) = |A|\cos\left(\omega t + \alpha\right)$$

$$B = \mathrm{Re}\left(Be^{j\omega t}\right) = |B|\cos\left(\omega t + \beta\right)$$

where A and B are the phasor quantities.

$$A = |A|e^{j\alpha}$$

$$B = |B|e^{j\beta}$$

Therefore,

$$AB = |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)$$

$$= \frac{1}{2}|A||B|\left[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\right]$$

Since A and B are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB,

$$AB_{average} = \frac{1}{T}\int_0^T AB\,dt$$

$$AB_{average} = \frac{1}{T}\int_0^T |A||B|cos(\omega t + \alpha)cos(\omega t + \beta)dt$$

$$AB_{average} = \frac{1}{2}|A||B|cos(\alpha - \beta)$$

For Phasors,

$$AB^* = |A|e^{j\alpha}|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$$

and $\mathrm{Re}(AB^*) = |A||B|\cos(\alpha - \beta)$, where * denotes complex conjugate.

$$AB_{average} = \frac{1}{2}Re\{AB^*\}$$

The instantaneous Poynting vector $\vec{P}(z,t) = \vec{E}(z,t) \times \vec{H}(z,t)$ can be expressed as

$$\vec{P} = \hat{x}(E_y(z,t)H_z(z,t) - E_z(z,t)H_y(z,t)) + \hat{y}(E_z(z,t)H_x(z,t) - E_x(z,t)H_z(z,t)) + \hat{z}(E_x(z,t)H_y(z,t) - E_y(z,t)H_x(z,t))$$

If we consider a plane electromagnetic wave propagating in +z direction which has only E_x and H_y, we can write:

$$\vec{P}_z = E_x(z,t)H_y(z,t)\hat{z}$$

$$\vec{P}_{z,average} = \frac{1}{2} Re\{E_x(z) H_y^*(z) \hat{z}\}$$

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration.

For a general case, we can write

$$\vec{P}_{z,average} = \frac{1}{2} Re\{\vec{E} \times \vec{H}^*\}$$

We can define a complex Poynting vector

$$\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^*$$

and time average of the instantaneous Poynting vector is given by

$$\vec{P}_{z,average} = Re\{\vec{S}\}$$