

Design a digital Butterworth filter satisfying the constraints

$$0.707 \leq |H(e^{j\omega})| \leq 1 \quad \text{for } 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(e^{j\omega})| \leq 0.2 \quad \text{for } \frac{3\pi}{4} \leq \omega \leq \pi$$

With $T=1$ sec using Impulse invariant method. [Nov/Dec-13]

Solution:

Given data:

Pass band attenuation $\alpha_p = 0.707$; Pass band frequency $\omega_p = \frac{\pi}{2}$;

Stop band attenuation $\alpha_s = 0.2$; Stop band frequency $\omega_s = \frac{3\pi}{4}$;

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_p = -20 \log \delta_1 = -20 \log(0.707) = 3.0116 \text{ dB}$

Stop band attenuation $\alpha_s = -20 \log \delta_2 = -20 \log(0.2) = 13.9794 \text{ dB}$

Step 2. Choose T and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\omega_p = \Omega_p T = \frac{\pi}{2} \text{ Rad / Sec}$$

$$\omega_s = \Omega_s T = \frac{3\pi}{4} \text{ Rad / Sec}$$

Step 3. To find order of the filter

$$N \geq \left[\frac{\log_{10} \sqrt{\left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right)}}{\log_{10} \left(\frac{\Omega_s}{\Omega_p} \right)} \right]$$

$$\begin{aligned}
 N &\geq \frac{\log \sqrt{\left(\frac{10^{0.1*3.01} - 1}{10^{0.1*13.97} - 1}\right)}}{\log \left(\frac{3\pi}{4}\right)} \\
 &\geq \frac{\log \sqrt{\left(\frac{0.9998}{23.945}\right)}}{\log(1.5)} \\
 &\geq \frac{\log(0.20433)}{\log(1.5)} \\
 &\geq \frac{-0.6896}{0.17609} \\
 &\geq 3.924
 \end{aligned}$$

Rounding the next higher value $N=4$

Step 4: The normalized transfer function

$$H_a(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

Step 5: Cut off frequency

$$\begin{aligned}
 \Omega_c &= \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} \\
 \Omega_c &= \frac{\frac{\pi}{2}}{(10^{0.1*3.01} - 1)^{\frac{1}{2*4}}} = \frac{\frac{\pi}{2}}{(0.9998)^{\frac{1}{8}}} = 1.57 \text{ Rad / Sec}
 \end{aligned}$$

Step 6: To find Transfer function of $H(s)$:

$$\begin{aligned}
 H(s) &= H_a(s) \Big|_{s \rightarrow \frac{s}{1.57}} \\
 H(s) &= \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)} \Big|_{s \rightarrow \frac{s}{1.57}}
 \end{aligned}$$

$$= \frac{1}{\left(\left(\frac{s}{1.57} \right)^2 + 0.76537 \left(\frac{s}{1.57} \right) + 1 \right) \left(\left(\frac{s}{1.57} \right)^2 + 1.8477 \left(\frac{s}{1.57} \right) + 1 \right)}$$

$$H(s) = \frac{(1.57)^4}{(s^2 + 1.202s + 2.465)(s^2 + 2.902s + 2.465)}$$

Step 7: Using partial fraction expansion, expand H(s) into

$$H(s) = \frac{A}{(s+1.45+j0.6)} + \frac{A^*}{(s+1.45-j0.6)} + \frac{B}{(s+0.6+1.45j)} + \frac{B^*}{(s+0.6-1.45j)}$$

To find A and A*:

$$A = H(s) \Big|_{s=-1.45-j0.6}$$

$$= (s+1.45-j0.6) \frac{(1.57)^4}{(s+1.45+j0.6)(s+1.45-j0.6)(s^2+2.902s+2.465)} \Big|_{s=-1.45-j0.6}$$

$$= \frac{(1.57)^4}{(s+1.45-j0.6)(s^2+1.202s+2.465)} \Big|_{s=-1.45-j0.6}$$

$$= \frac{(1.57)^4}{(-1.45-j0.6+1.45-j0.6)((-1.45-j0.6)^2+1.202(-1.45-j0.6)+2.465)}$$

$$= \frac{(1.57)^4}{-j(1.2)[1.7425+1.74j-1.7429-j0.7212+2.465]}$$

$$= \frac{(1.57)^4}{-j(1.2)[2.465+j1.0188]} = \frac{5.063}{1.0188-j2.465}$$

$$= \frac{5.063(1.0188-j2.465)}{7.114}$$

$$A = 0.7253 + j1.754; A^* = 0.7253 - j1.754$$

To find B and B*:

$$B = H(s) \Big|_{s=-0.6-j1.45}$$

$$= (s+0.6+j1.45) \frac{(1.57)^4}{(s+1.45+j0.6)(s+1.45-j0.6)(s+0.6+1.45j)(s+0.6-1.45j)} \Big|_{s=-0.6-j1.45}$$

$$= \frac{(1.57)^4}{(-0.6 - j1.45 + 1.45 + j0.6)(-0.6 - j1.45 + 1.45 - j0.6)(-0.6 - j1.45 + 0.6 - 1.45j)}$$

$s = -0.6 - j1.45$

$$(1.57)^4$$

$$= \frac{(1.57)^4}{(0.85 - j0.85)(0.85 - j0.85)(-2.9j)}$$

$$(1.57)^4$$

$$= \frac{(1.57)^4}{-j[-1.0187 - j2.468]}$$

$$2.095$$

$$= \frac{2.095}{-2.468 + j1.0187}$$

$$B = -0.7253 - j0.3; \quad B^* = -0.7253 + j0.3$$

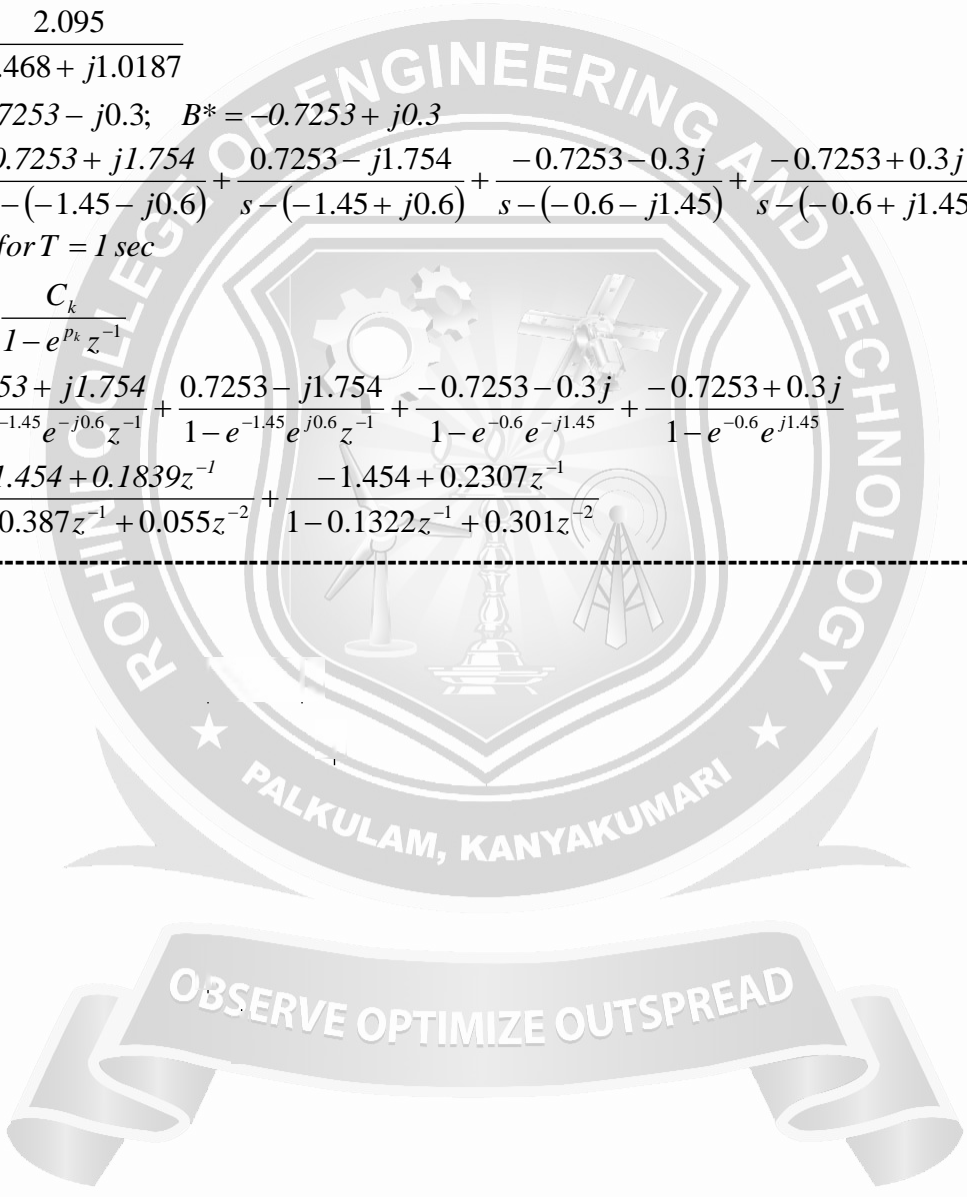
$$H(s) = \frac{0.7253 + j1.754}{s - (-1.45 - j0.6)} + \frac{0.7253 - j1.754}{s - (-1.45 + j0.6)} + \frac{-0.7253 - 0.3j}{s - (-0.6 - j1.45)} + \frac{-0.7253 + 0.3j}{s - (-0.6 + j1.45)}$$

we know for $T = 1 \text{ sec}$

$$H(z) = \sum_{k=1}^N \frac{C_k}{1 - e^{p_k} z^{-1}}$$

$$= \frac{0.7253 + j1.754}{1 - e^{-1.45} e^{-j0.6} z^{-1}} + \frac{0.7253 - j1.754}{1 - e^{-1.45} e^{j0.6} z^{-1}} + \frac{-0.7253 - 0.3j}{1 - e^{-0.6} e^{-j1.45} z^{-1}} + \frac{-0.7253 + 0.3j}{1 - e^{-0.6} e^{j1.45} z^{-1}}$$

$$H(z) = \frac{1.454 + 0.1839z^{-1}}{1 - 0.387z^{-1} + 0.055z^{-2}} + \frac{-1.454 + 0.2307z^{-1}}{1 - 0.1322z^{-1} + 0.301z^{-2}}$$



Design a chebyshev filter for the following specification using bilinear transformation.

$$0.8 \leq |H(e^{j\omega})| \leq 1 \quad 0 \leq \omega \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.2 \quad 0.6\pi \leq \omega \leq \pi.$$

Solution:

Given data:

Pass band attenuation $\alpha_p = 0.8$; Pass band frequency $\omega_p = 0.2\pi$;

Stop band attenuation $\alpha_s = 0.2$; Stops band frequency $\omega_s = 0.6\pi$;

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_p = -20 \log \delta_1 = -20 \log(0.8) = 1.938dB$

Stop band attenuation $\alpha_s = -20 \log \delta_2 = -20 \log(0.2) = 13.9794dB$

Step2. Choose T and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right) = 2 \tan\left(\frac{0.2\pi}{2}\right) = 0.649dB$$

$$\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right) = 2 \tan\left(\frac{0.6\pi}{2}\right) = 2.75dB$$

Step3. To find order of the filter

$$N \geq \frac{\cosh^{-1} \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\cosh^{-1} \left(\frac{\Omega_s}{\Omega_p} \right)}$$

$$\begin{aligned}
&\geq \frac{\cosh^{-1} \sqrt{\frac{10^{0.1 \cdot 13.97} - 1}{10^{0.1 \cdot 1.938} - 1}}}{\cosh^{-1} \left(\frac{2.75}{0.649} \right)} \\
&\geq \frac{\cosh^{-1} \sqrt{\frac{23.945}{0.562}}}{\cosh^{-1} \left(\frac{2.75}{0.649} \right)} \\
&\geq \frac{\cosh^{-1}(6.5273)}{\cosh^{-1}(4.2372)} \\
&\geq \frac{2.5632}{2.1228} \\
&\geq 1.207
\end{aligned}$$

Rounding the next higher integer value $N=2$

Step4. The poles of chebyshev filter can be determined by

$$S_k = a \cos \phi_k + jb \sin \phi_k, k = 0, 1, \dots, N$$

Where,

$$\phi_k = \left[\frac{(2k + N - 1)\pi}{2N} \right] \quad \text{And calculate a, b, } \varepsilon, \mu$$

$$\varepsilon = \sqrt{10^{0.1 \alpha p} - 1},$$

$$= \sqrt{10^{0.1 \cdot 1.938} - 1}$$

$$\varepsilon = 0.75$$

$$\mu = \varepsilon^{-1} + \left[\sqrt{1 + \varepsilon^{-2}} \right]$$

$$= (0.75)^{-1} + \left[\sqrt{1 + (0.75)^{-2}} \right]$$

$$\mu = 3$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right]$$

$$= 0.649 \left[\frac{(3)^{\frac{1}{2}} - (3)^{-\frac{1}{2}}}{2} \right]$$

$$a = 0.375$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

$$= 0.649 \left[\frac{(3)^{\frac{1}{2}} + (3)^{-\frac{1}{2}}}{2} \right]$$

$$b = 0.75$$

$$\phi_k = \left[\frac{(2k + N - 1)\pi}{2N} \right]; \quad k = 1, 2$$

$$\phi_1 = \left[\frac{(2(1) + 2 - 1)\pi}{2 * 2} \right] = \frac{3\pi}{4} = 135^\circ$$

$$\phi_2 = \left[\frac{(2(2) + 2 - 1)\pi}{2 * 2} \right] = \frac{5\pi}{4} = 225^\circ$$

$$S_k = a \cos \phi_k + j b \sin \phi_k, \quad k = 1, 2$$

for $k = 1$,

$$S_1 = 0.375 \cos \phi_1 + j(0.75) \sin \phi_1$$

$$= 0.375 \cos 135^\circ + j(0.75) \sin 135^\circ$$

$$S_1 = -0.265 + j0.53$$

for $k = 2$,

$$S_1 = 0.375 \cos \phi_2 + j(0.75) \sin \phi_2$$

$$= 0.375 \cos 225^\circ + j(0.75) \sin 225^\circ$$

$$S_1 = -0.265 - j0.53$$

Step.5 Find the denominator polynomial of the transfer function using above poles.

$$H(s) = \{S + 0.265 - j0.53\} \{S + 0.265 - j0.53\}$$

$$= \{(S + 0.265)^2 - (j0.53)^2\}$$

$$= (S + 0.265)^2 + (0.53)^2$$

$$= S^2 + 0.5306s + 0.3516$$

Step 6 : The numerator of the transfer function depends on the value of N.



If N is Even substitute $s=0$ in the denominator polynomial and divide the result by $\sqrt{1 + \epsilon^2}$. Find the value. This value is equal to numerator

$$= \frac{0.3516}{\sqrt{1 + \epsilon^2}} = \frac{0.3516}{\sqrt{1 + (0.75)^2}}$$

$$H(s) = 0.28$$

Step 7: The Transfer function is

$$H(s) = \frac{NM}{DM}$$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Step 8: Apply bilinear transformation with to obtain the digital filter

$$H(z) = H(s) \Big|_{s = \frac{2}{T} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)}$$

$$\begin{aligned}
 H(z) &= \frac{0.28}{s^2 + 0.5306s + 0.3516} \Big|_s = \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \\
 &= \frac{0.28}{s^2 + 0.5306s + 0.3516} \Big|_s = 2 \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \\
 &= \frac{0.28}{\left(2 \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \right)^2 + 0.5306 \left(2 \left(\frac{1-z^{-1}}{1+z^{-1}} \right) \right) + 0.3516} \\
 H(z) &= \frac{0.28(1+z^{-1})^2}{1-1.348z^{-1}+0.608z^{-2}}
 \end{aligned}$$

Design a chebyshev filter for the following specification using impulse invariance method.

$$0.8 \leq |H(e^{j\omega})| \leq 1 \quad 0 \leq \omega \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.2 \quad 0.6\pi \leq \omega \leq \pi. [May/June - 2016]$$

Solution:

Given data:

Pass band attenuation $\alpha_p = 0.8$; Pass band frequency $\omega_p = 0.2\pi$;

Stop band attenuation $\alpha_s = 0.2$; Stops band frequency $\omega_s = 0.6\pi$;

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_p = -20 \log \delta_1 = -20 \log(0.8) = 1.938dB$

Stop band attenuation $\alpha_s = -20 \log \delta_2 = -20 \log(0.2) = 13.9794dB$

Step2. Choose T and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_p = \frac{\omega_p}{T} = 0.2\pi \text{ Rad / Sec}$$

$$\Omega_s = \frac{\omega_s}{T} = 0.6\pi \text{ Rad / Sec}$$

Step3. To find order of the filter

$$N \geq \frac{\text{Cosh}^{-1} \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\text{Cosh}^{-1} \left(\frac{\Omega_s}{\Omega_p} \right)}$$

$$\begin{aligned} &\geq \frac{\cosh^{-1} \sqrt{\frac{10^{0.1 \cdot 13.97} - 1}{10^{0.1 \cdot 1.938} - 1}}}{\cosh^{-1} \left(\frac{0.6\pi}{0.2\pi} \right)} \\ &\geq \frac{\cosh^{-1} \sqrt{\frac{23.945}{0.562}}}{\cosh^{-1}(3)} \\ &\geq \frac{\cosh^{-1}(6.5273)}{\cosh^{-1}(3)} \\ &\geq \frac{2.5632}{1.7627} \\ &\geq 1.454 \end{aligned}$$

Rounding the next higher integer value $N=2$

Step4. The poles of chebyshev filter can be determined by

$$S_k = a \cos \phi_k + jb \sin \phi_k, k = 0, 1, \dots, N$$

Where,

$$\phi_k = \left[\frac{(2k + N - 1)\pi}{2N} \right] \quad \text{And calculate a, b, } \varepsilon, \mu$$

$$\varepsilon = \sqrt{10^{0.1\alpha p} - 1},$$

$$= \sqrt{10^{0.1 \cdot 1.938} - 1}$$

$$\varepsilon = 0.75$$

$$\mu = \varepsilon^{-1} + \left[\sqrt{1 + \varepsilon^{-2}} \right]$$

$$= (0.75)^{-1} + \left[\sqrt{1 + (0.75)^{-2}} \right]$$

$$\mu = 3$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right]$$

$$= 0.2\pi \left[\frac{(3)^{\frac{1}{2}} - (3)^{-\frac{1}{2}}}{2} \right]$$

$$a = 0.362$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

$$= 0.2\pi \left[\frac{(3)^{\frac{1}{2}} + (3)^{-\frac{1}{2}}}{2} \right]$$

$$b = 0.7255$$

$$\phi_k = \left[\frac{(2k + N - 1)\pi}{2N} \right]; \quad k = 1, 2$$

$$\phi_1 = \left[\frac{(2(1) + 2 - 1)\pi}{2 * 2} \right] = \frac{3\pi}{4} = 135^\circ$$

$$\phi_2 = \left[\frac{(2(2) + 2 - 1)\pi}{2 * 2} \right] = \frac{5\pi}{4} = 225^\circ$$

$$S_k = a \cos \phi_k + j b \sin \phi_k, \quad k = 1, 2$$

for $k = 1$,

$$S_1 = 0.362 \cos \phi_1 + j(0.7255) \sin \phi_1$$

$$= 0.362 \cos 135^\circ + j(0.7255) \sin 135^\circ$$

$$S_1 = -0.256 + j0.513$$

for $k = 2$,

$$S_1 = 0.362 \cos \phi_2 + j(0.7255) \sin \phi_2$$

$$= 0.362 \cos 225^\circ + j(0.7255) \sin 225^\circ$$

$$S_1 = -0.256 - j0.513$$

Step.5 Find the denominator polynomial of the transfer function using above poles.

$$H(s) = \{S + 0.256 - j0.513\} \{S + 0.256 - j0.513\}$$

$$= \{(S + 0.256)^2 - (j0.513)^2\}$$

$$= (S + 0.256)^2 + (0.513)^2$$

$$= S^2 + 0.513s + 0.33$$

Step 6 : The numerator of the transfer function depends on the value of N.



If N is Even substitute $s=0$ in the denominator polynomial and divide the result by $\sqrt{1 + \varepsilon^2}$ Find the value. This value is equal to numerator

$$= \frac{0.33}{\sqrt{1 + \varepsilon^2}} = \frac{0.33}{\sqrt{1 + (0.75)^2}}$$

$$H(s) = 0.264$$

Step 7: The Transfer function is

$$H(s) = \frac{NM}{DM}$$

$$H(s) = \frac{0.264}{s^2 + 0.513s + 0.33}$$

Step 8: Using partial fraction expansion, expand H(s) into

$$H(s) = \sum_{k=1}^2 \frac{A_k}{s - p_k} = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2}$$

$$\frac{0.264}{s^2 + 0.513s + 0.33} = \frac{A_1}{s - (-0.256 + j0.514)} + \frac{A_2}{s - (-0.256 - j0.514)}$$

$$= \frac{0.257j}{s - (-0.256 + j0.514)} - \frac{0.257j}{s - (-0.256 - j0.514)}$$

Step 9: Now transform analog poles $\{P_k\}$ into digital poles $\{e^{p_k T}\}$ to obtain the digital filter

$$\begin{aligned}
 H(z) &= \sum_{k=1}^N \frac{A_k}{1 - e^{p_k T} z^{-1}} \\
 &= \sum_{k=1}^2 \frac{A_k}{1 - e^{p_k T} z^{-1}} \\
 &= \frac{0.257j}{s - e^{-0.256T} e^{j0.513T} z^{-1}} - \frac{0.257j}{s - e^{-0.256T} e^{-j0.513T} z^{-1}} \\
 H(z) &= \frac{0.1954z^{-1}}{1 - 1.3483z^{-1} + 0.5987z^{-2}}
 \end{aligned}$$

