UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

3.2 ONE DIMENSIONAL HEAT EQUATION

One dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } \alpha^2 = \frac{k}{\rho c} = \frac{\text{Thermal conductivity}}{\text{density} \times \text{Specific heat capacity}}$$

$u(x,t) \rightarrow$ The temperature distribution at any point *x* from one end at time *t*.

The various Possible Solution of 1-D heat equation.

(i) $u(x,t) = (Ae^{px} + Be^{-px})Ce^{\alpha^2 p^2 t}$

(ii) $u(x,t) = (A\cos px + B\sin px)Ce^{-\alpha^2 p^2 t}$

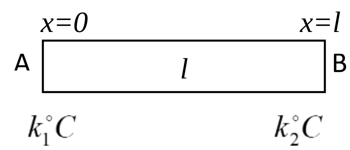
(iii) $u(x,t) = Ax + B$

The boundary and initial conditions.

i) $u(0,t) = k_1^{\circ}C$

ii) $u(l,t) = k_2^{\circ}C$

iii) $u(x,0) = f(x)$



The correct solution is $u(x,t) = \left(A\cos px + B\sin px\right)Ce^{-\alpha^2 p^2 t}$

The steady state solution in 1-D heat equation:

Solution:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad ------(1)$$

In steady state *t=0* then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $\boxed{u(x) = Ax + B}$

A rod of length *l* has its ends *A* and *B* are kept at $0^\circ C$ and $100^\circ C$ until steady state condition prevail. If the temperature at *B* is reduced suddenly to $0^\circ C$ and kept so while that of *A* is maintained. Find the temperature $u(x,t)$ at a distance x from *A* and at time *t*.

Solution:

The 1-D heat equation is $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \dfrac{k}{\rho c}$

To find steady state solution $u(x,0) = u(x)$

In steady state *t=0* then $\dfrac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $\boxed{u(x) = Ax + B}$ ----(1)

The boundary conditions are i) $u(0) = 0^\circ C$ ii) $u(l) = 100^\circ C$

Applying condn (i) in (1)

$$(1) \Rightarrow u(0) = 0 + B \Rightarrow \boxed{B = 0}$$

Sub B in (1)

$$u(x) = Ax \text{ ----}(2)$$

Applying condn (ii) in (2)

$$u(l) = Al \Rightarrow 100 = Al \Rightarrow \boxed{A = \frac{100}{l}}$$

Sub A in (2)

$$\boxed{u(x) = \frac{100x}{l}}$$

The boundary and initial conditions are

i) $u(0,t) = 0^\circ C$

ii) $u(l,t)=100^{\circ}C$

iii) $u(x,0)=f(x)=\dfrac{100x}{l},\ 0\le x\le l$

The correct solution is

$$u(x,t)=\left(A\cos px+B\sin px\right)Ce^{-\alpha^2p^2t}\ \text{-----}(1)$$

Apply condn. (i) in (1)

$$u(0,t)=\left(A\cos 0+\cancel{B\sin 0}\right)Ce^{-\alpha^2p^2t}$$

$$0=ACe^{-\alpha^2p^2t}$$

Here $C\neq 0,\ e^{-\alpha^2p^2t}\neq 0\ \therefore \boxed{A=0}$

Sub A in (1)

$$u(x,t)=\left(B\sin px\right)Ce^{-\alpha^2p^2t}\ \text{-----}(2)$$

Apply condn. (ii) in (2)

$$u(l,t)=\left(B\sin pl\right)Ce^{-\alpha^2p^2t}$$

$$0=\left(B\sin pl\right)Ce^{-\alpha^2p^2t}$$

Here $B\neq 0, C\neq 0, e^{-\alpha^2p^2t}\neq 0\ \therefore \boxed{\sin pl=0}$

$$\sin pl=\sin n\pi\ \Rightarrow pl=n\pi\ \Rightarrow \boxed{p=\frac{n\pi}{l}}$$

Sub p in (2)

$$u(x,t)=\left(B\sin\frac{n\pi x}{l}\right)Ce^{-\frac{\alpha^2n^2\pi^2t}{l^2}}$$

$$u(x,t)=b_1\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2n^2\pi^2t}{l^2}}\quad BC=b_1\text{(say)}$$

The most general solution is

$$\boxed{u(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}} \quad ------(3)$$

Apply condn (iii) in (3)

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-0}$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \because e^{-0}=1$$

This is Fourier sine series of *f(x)* in *(0,l)*

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}dx$$

$$=\frac{200}{l^2}\int_0^l x\sin\frac{n\pi x}{l}dx$$

$$=\frac{200}{l^2}\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$=\frac{200}{l^2}\left(\frac{-l}{n\pi}\right)\left[x\cos\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{-200}{ln\pi}\left[l\cos n\pi-0\right]$$

$$=\frac{-200(-1)^n}{n\pi}$$

$$\boxed{b_n=\frac{200}{n\pi}(-1)^{n+1}}$$

Sub b_n in (3)

$$u(x,t)=\sum_{n=1}^{\infty}\frac{200}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2n^2\pi^2t}{l^2}}$$

$$\boxed{u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2n^2\pi^2t}{l^2}}}$$

This is the required temperature.

The ends A and B of a rod *l* cm long have their temperatures kept at $30^\circ C$ and $80^\circ C$, until steady state conditions prevail. The temperature of the end B is suddenly reduced to $60^\circ C$ and that of A is increased to $40^\circ C$. Find the steady state temperature distribution in the rod after time t.

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ where $\alpha^2=\frac{k}{\rho c}$

<u>To find steady state solution 1</u> $u(x,0)=u(x)$

In steady state *t=0* then $\frac{\partial u}{\partial t}=0$

$$\alpha^2\frac{\partial^2 u}{\partial x^2}=0 \Rightarrow \frac{\partial^2 u}{\partial x^2}=0 \Rightarrow \frac{d^2u}{dx^2}=0$$

Integrating twice we get $\boxed{u(x)=Ax+B}----(1)$

The boundary conditions are i) $u(0)=30^\circ C$ ii) $u(l)=80^\circ C$

Applying condn (i) in (1)

$$(1)\Rightarrow u(0)=0+B\Rightarrow \boxed{B=30}$$

Sub B in (1)

$$u(x)=Ax+30----(2)$$

Applying condn (ii) in (2)

$$u(l)=Al+30\Rightarrow 80=Al+30 \Rightarrow \boxed{A=\frac{50}{l}}$$

Sub A in (2)

$$u(x)=\frac{50x}{l}+30=f(x)$$

This $u(x)$ will be treated as the initial conditions $u(x,0)=f(x)$

<u>To find steady state solution 2</u> $u(x,0)=u(x)$

Integrating twice we get $\boxed{u_t(x)=Ax+B}$ ----(3)

The boundary conditions are i) $u_t(0)=40^\circ C$ ii) $u_t(l)=60^\circ C$

Applying condn (i) in (3)

$$(3)\Rightarrow u_t(0)=0+B\Rightarrow \boxed{B=40}$$

Sub B in (1)

$$u_t(x)=Ax+40 ----(4)$$

Applying condn (ii) in (4)

$$u_t(l)=Al+40\Rightarrow 60=Al+40 \Rightarrow \boxed{A=\frac{20}{l}}$$

Sub A in (2)

$$u_t(x)=\frac{20x}{l}+40$$

This $u_t(x)$ will be treated as the transient state temperature.

The required temperature is

$$u(x,t)=u_t(x,0)+\left(A\cos px+B\sin px\right)Ce^{-\alpha^2p^2t}$$

$$u(x,t)=\frac{20x}{l}+40+\left(A\cos px+B\sin px\right)Ce^{-\alpha^2p^2t} ------(5)$$

The boundary and initial conditions are

i) $u(0,t)=40^\circ C$

ii) $u(l,t)=60^\circ C$

iii) $u(x,0)=f(x)=\frac{50x}{l}+30,\ \ 0\le x\le l$

Apply condn (i) in (5)

$$u(0,t)=0+40+\left(A\cos 0+B\sin 0\right)Ce^{-\alpha^2p^2t}$$

$$40=0+40+\left(A\cos 0+\cancel{B\sin 0}\right)Ce^{-\alpha^2p^2t}$$

$$0=ACe^{-\alpha^2p^2t}$$

This $C\ne 0, e^{-\alpha^2p^2t}\ne 0\ \therefore\ \boxed{A=0}$

Sub A in (5)

$$u(x,t)=\frac{20x}{l}+40+\left(B\sin px\right)Ce^{-\alpha^2p^2t}\ -----(6)$$

Apply condn (ii) in (6)

$$u(l,t)=20+40+\left(B\sin pl\right)Ce^{-\alpha^2p^2t}$$

$$60=20+40+\left(B\sin pl\right)Ce^{-\alpha^2p^2t}$$

$$0=\left(B\sin pl\right)Ce^{-\alpha^2p^2t}$$

$$B\ne 0, C\ne 0, e^{-\alpha^2p^2t}\ne 0\ \ \therefore \sin pl=0$$

$$\sin pl=\sin n\pi\ \Rightarrow\ pl=n\pi\ \Rightarrow\ \boxed{p=\frac{n\pi}{l}}$$

Sub p in (6)

$$u(x,t)=\frac{20x}{l}+40+\left(B\sin\frac{n\pi x}{l}\right)Ce^{-\frac{\alpha^2n^2\pi^2t}{l}}$$

$$u(x,t)=\frac{20x}{l}+40+b_1\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2n^2\pi^2t}{l}}\qquad \because BC=b_1$$

The most general solution is

$$u(x,t)=\frac{20x}{l}+40+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} \quad ------(7)$$

Apply condn (iii) in (7)

$$u(x,0)=\frac{20x}{l}+40+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-0}$$

$$\frac{50x}{l}+30=\frac{20x}{l}+40+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{30x}{l}-10=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

To find b_n:

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l\left(\frac{30x}{l}-10\right)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[\left(\frac{30x}{l}-10\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-\left(\frac{30}{l}\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[\left(\frac{-l}{n\pi}\right)\left(\frac{30x}{l}-10\right)\cos\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{-2}{n\pi}\left[(20)\cos n\pi+10\right]$$

$$\boxed{b_n=\frac{-20}{n\pi}\left[2(-1)^n+1\right]}$$

Sub b_n in (7)

$$u(x,t)=\frac{20x}{l}+40+\sum_{n=1}^{\infty}\frac{-20}{n\pi}\left[2(-1)^n+1\right]\sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$

$$u(x,t)=\frac{20x}{l}+40-\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[2(-1)^n+1\right]\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2 n^2\pi^2 t}{l}}$$