

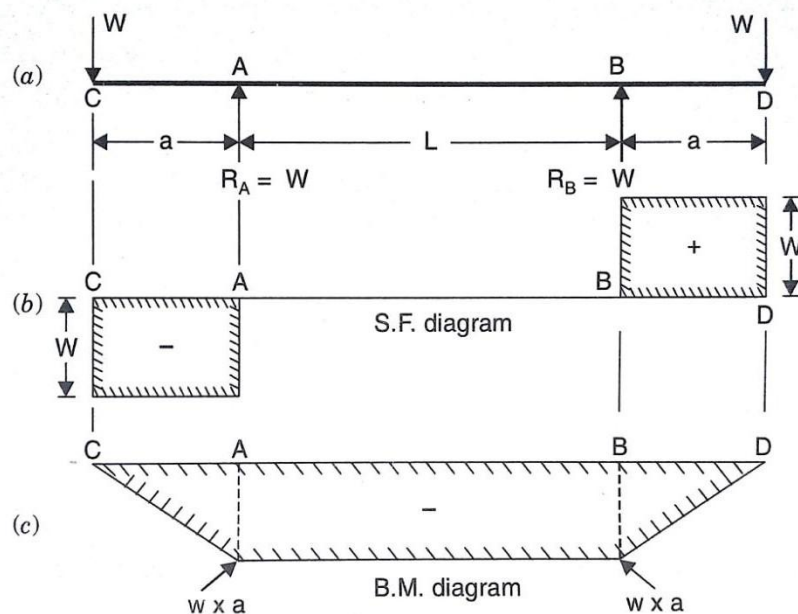
3.5 BENDING AND SHEAR STRESS IN BEAM

BENDING STRESS IN BEAM

When some external load acts on a beam, the shear force and bending moments are set up at all sections of the beam. Due to the shear force and bending moment, the beam undergoes certain deformation. The stresses introduced by bending moment is known as bending stresses.

PURE BENDING OR SIMPLE BENDING

If a length of a beam is subjected to a constant bending moment and no shear force (i.e., zero shear force), then the stresses will be set up in that length of the beam due to B.M. only and that length of beam is known as pure bending or simple bending. The stresses set up in that length of beam are known as bending stresses.



A beam simply supported at A and B and overhanging by same length at each support. A point load W is applied at each end of the overhanging portion. It is clear that there is no shear force between A and B but the B.M. between A and B is known as pure bending or simple bending.

THEORY OF SIMPLE BENDING WITH ASSUMPTIONS MADE

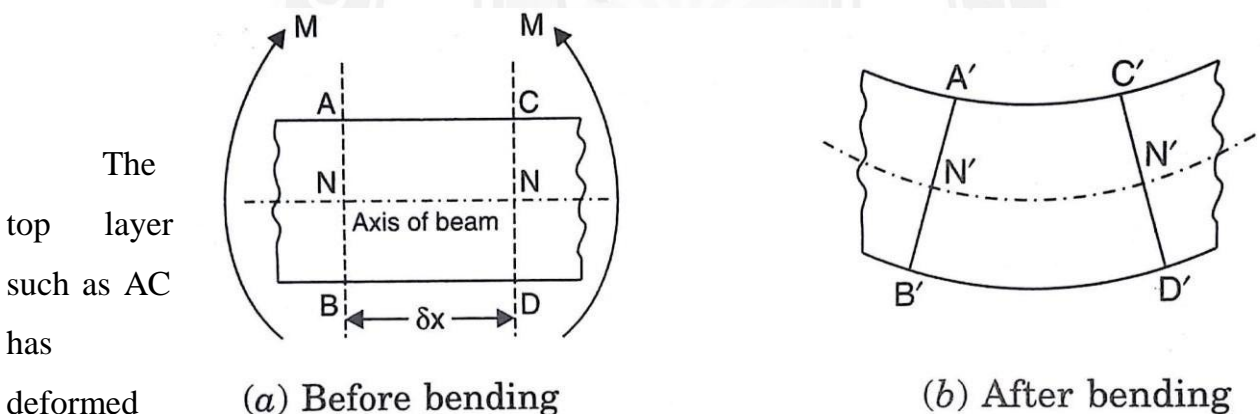
Assumptions made in the theory of simple bending. The following are the important assumptions:

1. The material of the beam is homogeneous and isotropic.

2. The value of young's modulus of elasticity is the same in tension and compression.
3. The transverse sections which were plane before bending, remain plane after bending also.
4. The beam is initially straight and all longitudinal filaments bend into circular areas with a common centre of curvature.
5. The radius of curvature is large compared with the dimensions of the cross section.
6. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.

Theory of Simple Bending

The fig. shows A part of a beam is subjected to simple bending. Consider a small length δx of this part of beam. Consider two sections AB and CD which are normal to the axis of the beam N-N. Due to the action of the bending moment, the part of length δx will be deformed.



to the shape $A'B'$. This layer has been elongated. It is clear that some of the layers have been shortened while some of them are elongated. At a level between the top and bottom of the beam there will be a layer which is neither shortened nor elongated. This layer is known as neutral layer or neutral surface.

This layer is $N'-N'$. The line of intersection of the neutral layer on a cross section of the beam is known as the neutral axis.

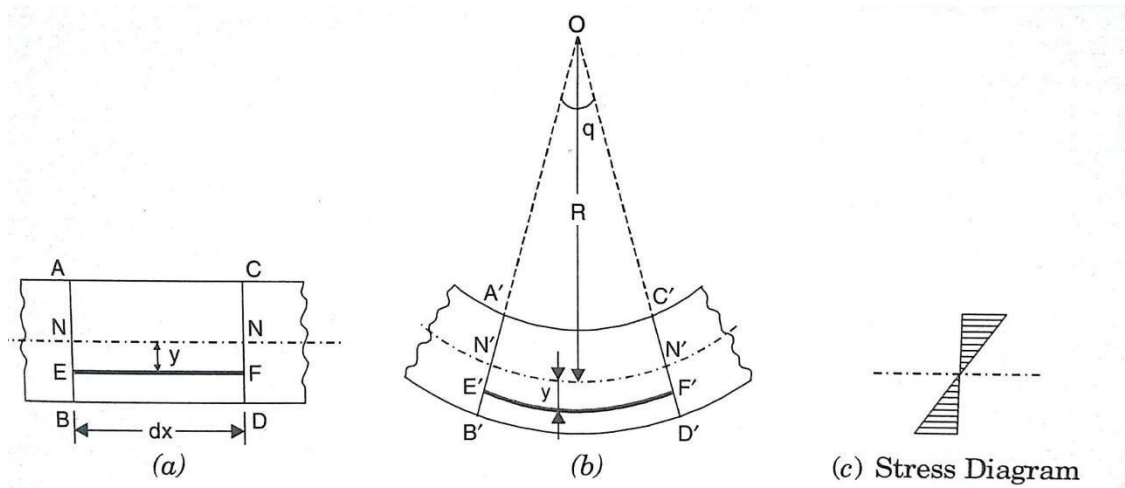
Hence the amount by which a layer increases or decreases in length, depends upon the position of the layer with respect to N-N. This theory of bending is known as theory of simple bending.

EXPRESSION FOR BENDING STRESS

A small length δx of a beam subjected to a simple bending. Due to the action of bending, the part of length δx will be deformed. Let $A'B'$ and $C'D'$ meet at O.

Let R = Radius of neutral layer $N'N'$

θ = Angle subtended at O by $A'B'$ and $C'D'$ produced.



Strain Variation along the Depth of Beam

Consider a layer EF at a distance y below the neutral layer NN . After bending this layer will be elongated to $E'F'$.

Original length of layer $EF = dx$.

Also length of neutral layer $NN = dx$.

After bending, the length of neutral layer $N'N'$ will remain unchanged. But the length of layer $E'F'$ will increase.

$$N'N' = NN = dx$$

$$N'N' = R \times \theta$$

$$E'F' = (R + y) \times \theta$$

$$\text{But } N'N' = NN = dx$$

$$\text{Hence } dx = R \times \theta$$

$$\therefore \text{Increase in length of layer } EF = E'F' - EF = (R + y) \times \theta - R \times \theta = y \times \theta$$

$$\begin{aligned} \therefore \text{Strain in layer } EF &= \frac{\text{Increase in length}}{\text{original length}} \\ &= y \times \frac{\theta}{dx} = \frac{y \times \theta}{R \times \theta} = \frac{y}{R} \end{aligned}$$

Stress Variation

Let σ = stress in layer EF

E = young's modulus of beam

E = stress in layer EF / strain in layer EF

$$= \sigma / (y/R)$$

$$\sigma = E \times \frac{y}{R} = \frac{E}{R} \times y$$

It can also be written as $\frac{\sigma}{y} = \frac{E}{R}$

NEUTRAL AXIS AND MOMENT OF RESISTANCE

The neutral axis of any transverse section of beam is defined as the line of intersection of the neutral layer with the transverse section. It is written as NA.

The stress at a distance y from the neutral axis is given by

$$\sigma = \frac{E}{R} \times y$$

Let dA = Area of layer

Force on layer = stress on layer \times area of layer

$$= \sigma \times dA$$

$$= \frac{E}{R} \times y \times dA$$

$$\text{Total force on beam section} = \int \frac{E \times y}{R} \times dA$$

$$= \frac{E}{R} \int y \times dA$$

But for pure bending there is no force on the section of beam

$$\therefore \frac{E}{R} \int y \times dA = 0$$

$$\int y \times dA = 0$$

Hence $\int y \times dA$ represents the moment of entire area of the section about neutral axis.

Moment of Resistance

Due to pure bending the layers above the NA are subjected to compressive stresses whereas the layers below the NA are subjected to tensile stresses. Due to these stresses forces acts on the layers.

The force on the layer at a distance y from neutral axis.

$$\text{Force on layer} = \frac{E \times y}{R} \times dA$$

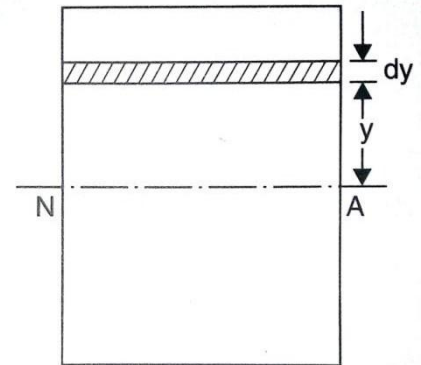
Moment of this force about NA

$$= \text{force on layer} \times y$$

$$= \frac{E \times y}{R} \times dA \times y$$

$$= \frac{E}{R} \times y^2 \times dA$$

$$\text{Total moment of the forces on the section of beam} = \int \frac{E}{R} \times y^2 \times dA$$



$$= \frac{E}{R} \int y^2 \times dA$$

Let M = External moment applied on the beam section.

$$M = \frac{E}{R} \int y^2 \times dA$$

But the expression $\int y^2 \times dA$ represent the moment of inertia of the area of the section about the neutral axis I

Then
$$M = \frac{E \times I}{R}$$

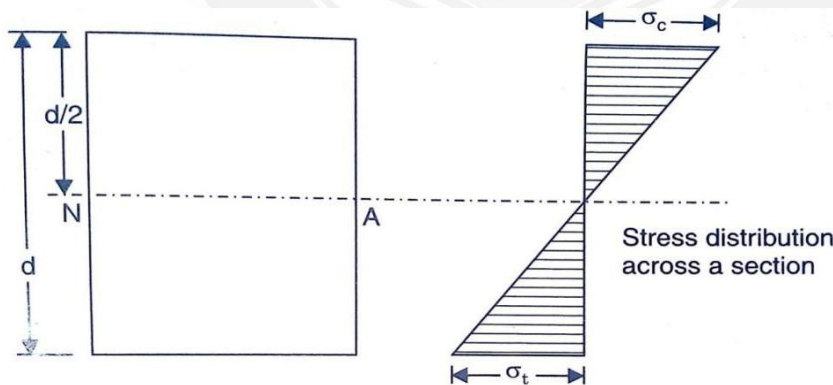
We know that
$$\frac{\sigma}{y} = \frac{E}{R}$$

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

It is known as the bending equation.

BENDING STRESSES IN SYMMETRICAL SECTIONS

The neutral axis of a symmetrical section lies at a distance of $d/2$ from the outermost layer of the section where d is the diameter or depth. There is no stress at the neutral axis. But the stress at a point is directly proportional to its distance from the neutral axis. The maximum shear stress takes place at the outermost layer. For a simply supported beam, there is a compressive stress above the neutral axis and a tensile stress below it. If we plot these stresses, we will get a figure shown below.



Problem 3.1 A steel plate of width 120mm and of thickness 20mm is bent into a circular arc of radius 10m. Determine the maximum stress induced and the bending moment which will provide the maximum stress. Take $E = 2 \times 10^5 \text{ N/mm}^2$.

Given:

Width $b = 120 \text{ mm}$

Thickness $t = 20 \text{ mm}$

$$\text{Moment of inertia } I = \frac{bt^3}{12} = 120 \times \frac{20^3}{12} = 8 \times 10^4 \text{ mm}^4$$

Radius of curvature $R = 10 \text{ m} = 10 \times 10^3 \text{ mm}$

Young's modulus $E = 2 \times 10^5 \text{ N/mm}^2$

Solution:

Wkt

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$y_{\max} = \frac{t}{2} = \frac{20}{2} = 10\text{mm}$$

$$\begin{aligned}\sigma_{\max} &= \frac{E}{R} \times y_{\max} \\ &= \frac{2 \times 10^5}{10 \times 10^3} \times 10 \\ &= 200 \text{ N/mm}^2\end{aligned}$$

Wkt

$$\frac{M}{I} = \frac{E}{R}$$

$$\begin{aligned}M &= \frac{E}{R} \times I \\ &= \frac{2 \times 10^5}{10 \times 10^3} \times 8 \times 10^4 \\ &= 16 \times 10^5 \text{ Nmm} = \mathbf{1.6 \text{ KNm}}\end{aligned}$$

Problem 3.2 Calculate the maximum stress induced in a cast iron pipe of external diameter 40mm of internal diameter 20mm and of length 4m when the pipe is supported on its ends and carries a point load of 80N at its centre.

Given

Ext dia. $D = 40\text{mm}$

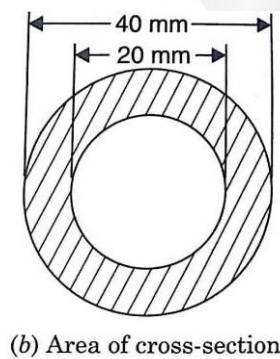
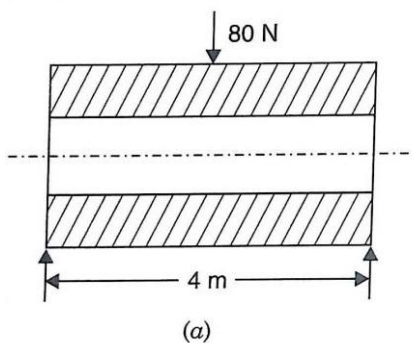
Int dia. $d = 20\text{mm}$

Length $L = 4\text{m} = 4 \times 1000 = 4000\text{mm}$

Point load $W = 80 \text{ N}$

Solution:

In this case of simply supported beam carrying a point load at the centre, the maximum bending moment is at the centre of the beam.



$$\text{Max bending moment BM} = \frac{W \times L}{4}$$

$$= 80 \times 4000 / 4 = 8 \times 10^4 \text{ Nmm}$$

$$\begin{aligned} \text{Moment of inertia of hollow pipe } I &= \frac{\pi}{64} (D^4 - d^4) = \frac{\pi}{64} (40^4 - 20^4) \\ &= 117809.7 \text{ mm}^4 \end{aligned}$$

Now

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$y_{\max} = D/2 = 40/2 = 20 \text{ mm}$$

$$\frac{M}{I} = \frac{\sigma_{\max}}{y_{\max}}$$

$$\begin{aligned} \sigma_{\max} &= \frac{M}{I} \times y_{\max} \\ &= 8 \times 10^4 \times \frac{20}{117809.7} \\ &= 13.58 \text{ N/mm}^2 \end{aligned}$$

SECTION MODULUS

Section modulus is defined as the ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis.

$$Z = \frac{I}{y_{\max}}$$

Where I = moment of inertia about neutral axis

y_{\max} = distance of the outermost layer from the neutral axis

Wkt

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{M}{I} = \frac{\sigma_{\max}}{y_{\max}}$$

$$M = \sigma_{\max} \times \frac{I}{y_{\max}}$$

But

$$\frac{I}{y_{\max}} = Z$$

\therefore

$$M = \sigma_{\max} \times Z$$

SECTION MODULUS FOR VARIOUS SHAPES OR BEAM SECTIONS

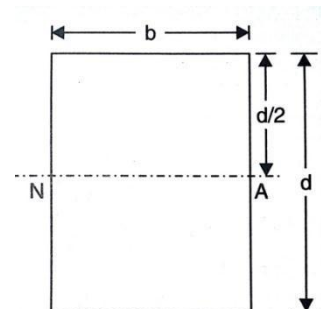
1. Rectangular Section

$$\text{Moment of inertia } I = bd^3/12$$

Distance of outermost layer from NA

$$y_{\max} = d/2$$

$$\therefore \text{Section modulus } Z = \frac{I}{y_{\max}} = \frac{bd^3/12}{d/2} = \frac{bd^2}{6}$$



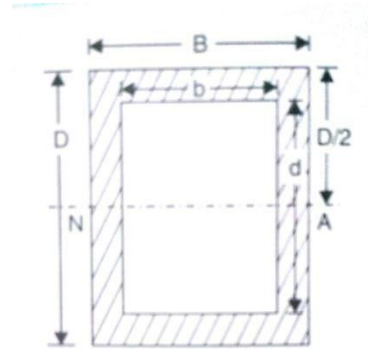
2. Hollow Rectangular Section

$$I = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$= \frac{1}{12} (BD^3 - bd^3)$$

$$y_{max} = D/2$$

$$\begin{aligned} \therefore Z &= \frac{I}{y_{max}} \\ &= \frac{\frac{1}{12} (BD^3 - bd^3)}{D/2} \\ &= \frac{1}{6D} (BD^3 - bd^3) \end{aligned}$$



3. Circular Section

$$I = \frac{\pi}{64} d^4 \quad \text{and} \quad y_{max} = d/2$$

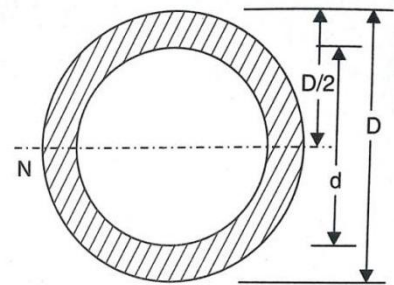
$$\therefore Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64} d^4}{d/2} = \frac{\pi}{32} d^3$$

4. Hollow Circular Section

$$I = \frac{\pi}{64} (D^4 - d^4)$$

$$y_{max} = D/2$$

$$\begin{aligned} \therefore Z &= I/y_{max} \\ &= \frac{32}{D} (D^4 - d^4) \end{aligned}$$



Problem 3.4. A cantilever of length 2m fails when a load of 2kN is applied at a free end. If the section of the beam is 40mm×60mm. Find the stress at the failure.

Given

Length	$L = 2\text{m}$
Load	$W = 2\text{Kn}$
Width	$b = 40\text{mm}$
Depth	$d = 60\text{mm}$

Solution:

Section modulus of rectangular section $Z = bd^2/6$

$$= 40 \times 60^2 / 6$$

$$= 24000 \text{mm}^3$$

Max bending moment for cantilever at fixed end $M = W \times L$

$$= 2000 \times 2 \times 10^3$$

$$= 4 \times 10^6 \text{Nm}$$

$$M = Z \times \sigma_{max}$$

$$\sigma_{max} = M/Z = 166.67 \text{ N/mm}^2$$

Problem 3.5. A rectangular beam 200mm deep and 300 mm wide is simply supported over a span of 8m. What uniformly distributed load per metre the beam is carrying if the bending stress is not to exceed 120N/mm^2

Given

Depth $d = 200\text{mm}$

Width $b = 300\text{mm}$

Length $L = 8\text{m}$

Max bending stress $\sigma_{\max} = 120\text{N/mm}^2$

Section modulus for rectangular section $Z = bd^2/6$

$$= 300 \times 200 \times 200 / 6$$

$$= 2000000\text{mm}^3$$

Max bending moment for simply supported beam carrying uniform load M

$$= w \times L^2 / 8$$

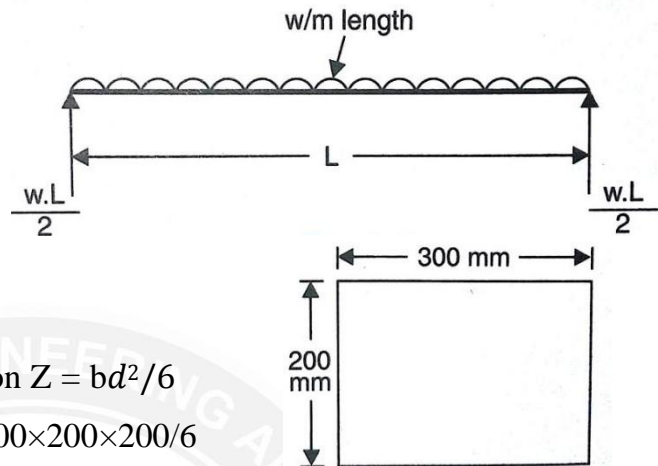
$$= 8w \text{ Nm} = 8w \times 1000\text{Nmm}$$

$$= 8000w \text{ Nmm}$$

$$M = Z \times \sigma_{\max}$$

$$8000w = 120 \times 2000000$$

$$w = 30\text{KN/m}$$



Problem 3.6: A rectangular beam 300 mm deep is simply supported over a span of 4m. Determine the uniformly distributed load per metre which the beam may carry if the bending stress should not exceed 120N/mm^2 . Take I

$$= 8 \times 10^6\text{mm}^4$$

Given

Depth $d = 300\text{mm}$

Span $L = 4\text{m}$

Max bending stress $M = 120\text{N/mm}^2$

Moment of inertia $I = 8 \times 10^6\text{mm}^4$

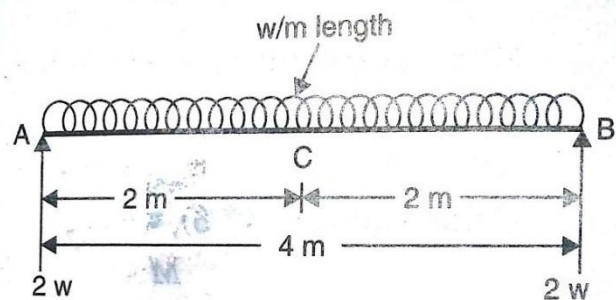
Max BM $= 2w \times 2 - 2w \times 1$

$$= 4w - 2w$$

$$= 2w\text{Nm}$$

$$= 2w \times 1000 = 2000w \text{ Nmm}$$

$$M = 2000w \text{ Nmm}$$



$$M = \sigma_{max} \times Z$$

$$Z = I/y_{max} = 8 \times 10^6 / 150$$

$$\therefore 2000w = 120 \times Z \quad w = 3200 \text{ N/m}$$

Problem 3.7: A square beam 20mm×20mm in section and 2m long is supported at the ends. The beam fails when a point load of 400N is applied at the centre of the beam. What uniformly distributed load per metre length will break a cantilever of the same material 40mm wide 60mm deep and 3m long.

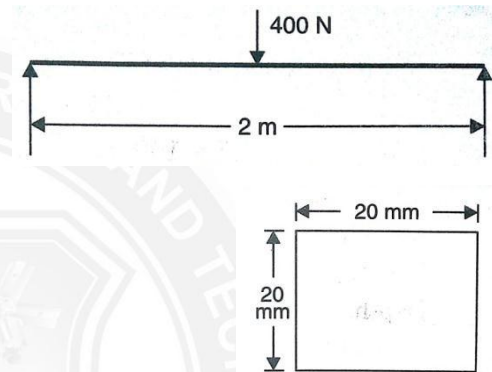
Given

Depth $d = 20\text{mm}$

Width $b = 20\text{mm}$

Length $L = 2\text{m}$

Point load $W = 400\text{N}$



Solution:

Section modulus $Z = bd^3/6$

$$= 4000/3 \text{ mm}^3$$

Max bending moment $M = w \times L/4 = 400 \times 2/4$

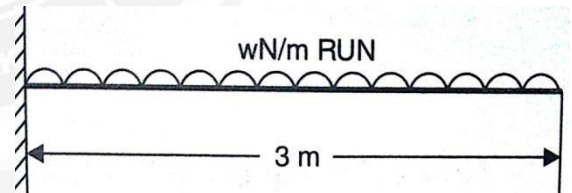
$$= 200\text{Nm}$$

$$= 200 \times 1000 = 200000 \text{ Nmm}$$

$$M = Z \times \sigma_{max}$$

$$20000 = \sigma_{max} \times (4000/3)$$

$$\sigma_{max} = 150 \text{ N/mm}^2$$



Let $w =$ uniformly distributed load per m run

Let $b = 40 \text{ mm}$

$d = 60 \text{ mm}$

$L = 3\text{m}$

Section modulus $Z = bd^2/6 = 40 \times 60^2/6 = 24000 \text{ mm}^3$

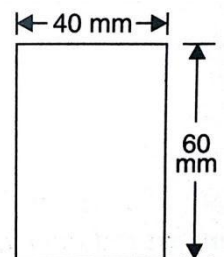
Max bending moment $= wL^2/2$

$$M = 4.5 \times 1000w \text{ Nmm}$$

Wkt $M = \sigma_{max} \times Z$

$$4.5 \times 1000w = 150 \times 24000$$

$$w = 800 \text{ N/m}$$



Problem 3.8: A beam is simply supported and carries a uniformly distributed load of 40kN/m run over the whole span. The section of beam is rectangular having depth as 500mm. If the max stress in the material of beam is 120N/mm^2 and a moment of inertia of the section is $7 \times 10^8\text{mm}^4$. Find the span of beam

Given

U.D.L $w = 40\text{kN/m}$

Depth $d = 500\text{mm}$

Max stress $\sigma_{max} = 120\text{ N/mm}^2$

M.O.I $I = 7 \times 10^8\text{mm}^4$

Section modulus $Z = I/Y_{max}$

$$y_{max} = d/2 = 500/2 = 250\text{ mm}$$

$$Z = 7 \times 10^8 / 250 = 28 \times 10^5\text{mm}^3$$

Max B.M for a simply supported beam carrying U.D.L $= w \times L^2/8$

$$M = 40000 \times L^2/8$$

$$= 5000L^2 \times 1000\text{Nmm}$$

Wkt $M = \sigma_{max} \times Z$

$$5000 \times 1000 \times L^2 = 120 \times 28 \times 10^5$$

$$L = \sqrt{2.4 \times 28} = 8.197\text{m}$$

Problem 3.9: A timber beam of rectangular section is to support a load of 20 kN uniformly distributed over a span of 3.6 m when beam is simply supported. If the depth of the section is to be twice the breadth and the stress in the timber is not to exceed 7N/mm^2 . Find the dimensions of how would you modify the cross section of the beam if it carries a concentrated load of 20 k N placed at the centre with the same ratio of breadth to depth.

Given

Total load $W = 20\text{ k N}$

Span $L = 3.6\text{m}$

Max stress $\sigma_{max} = 7\text{N/mm}^2$

Depth $d = 2b\text{ mm}$

Section modulus $Z = bd^2/6$

$$Z = b \times (2b) \times (2b) / 6 = \frac{2b^3}{3}\text{mm}^3$$

Max B.M $= wL^2/8 \text{ or } WL/8$

$$M = WL/8 = 9000\text{ Nm} = 9000 \times 1000\text{Nmm}$$

$$W k t \quad M = Z \times \sigma_{max}$$

$$9000 \times 1000 = 7 \times \frac{2b^3}{3}$$

$$b^3 = 1.92857 \times 10^6$$

$$b = 124.5 \text{ mm}$$

$$d = 2b = 2 \times 124.5 = 249 \text{ mm}$$

B.M is maximum at the centre and is equal to $WL/4$

$$M = WL/4 = 20000 \times 3.6/4 = 18000 \text{ N m}$$

$$= 18000 \times 1000 \text{ Nmm}$$

$$\sigma_{max} = 7 \text{ N/mm}^2$$

$$Z = 2b^3/3$$

$$\text{We get} \quad M = Z \times \sigma_{max}$$

$$18000 \times 1000 = 7 \times 2b^3/3$$

$$b = \mathbf{156.82 \text{ mm}}$$

$$d = 2 \times 156.82 = \mathbf{313.64 \text{ mm}}$$

Problem 3.10: A timber beam of rectangular section of length 8m is simply supported. The beam carries a U.D.L of 12kN/m run over the entire length and a point load of 10k N at 3m from the left support. If the depth is two times the width and the stress in the timber is not to exceed 8 N/mm^2 . Find the suitable dimensions of the section.

Given

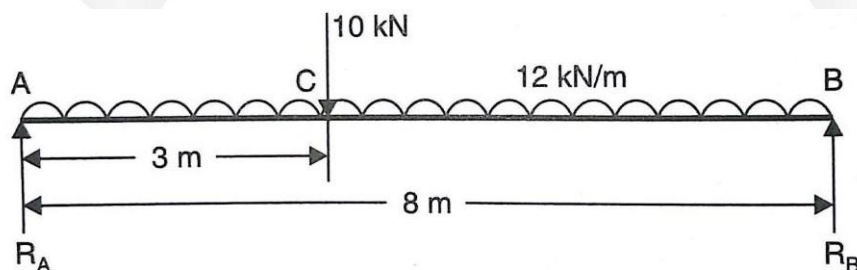
$$\text{Length} \quad L = 8 \text{ m}$$

$$\text{UDL} \quad w = 12000 \text{ N/m}$$

$$\text{Point load} \quad W = 10000 \text{ N}$$

$$\text{Depth of beam} \quad d = 2b$$

$$\sigma_{max} = 8 \text{ N/mm}^2$$



Solution:

Taking moment about A

$$\text{We get,} \quad R_B \times 8 = [(12000 \times 8) \times \frac{8}{2}] + (10000 \times 3) \text{ scoop}$$

$$R_B = 51750 \text{ N}$$

$$R_A = \text{total load} - R_B$$

$$= (12000 \times 8) + 10000 - 5175 = 54250 \text{ N}$$

$$\text{S.F at B} = -R_B = -51750 \text{ N}$$

$$\text{S.F at C} = -51750 + (12000 \times 5) = +8250 \text{ N} \quad (\text{without PL})$$

$$\text{S.F at C} = -51750 + (12000 \times 5) + 10000 = -18250 \text{ N} \quad (\text{with PL})$$

$$\text{S.F at A} = +R_A = +54250 \text{ N}$$

Let SF is zero at x metre from B

$$\text{Equating } (12000 \times x) - R_B = 0$$

$$(12000 \times x) - 51750 = 0$$

$$\therefore x = 4.3125 \text{ m}$$

\therefore Max BM will occur at 4.3125m from B

$$\begin{aligned} \therefore \text{Max BM} \quad M &= R_B \times 4.3125 - 12000 \times 4.3125 \times \frac{4.3125}{2} \\ &= 111585.9375 \times 10^3 \text{ Nmm} \end{aligned}$$

$$\text{Section modulus for rectangular beam } Z = \frac{bd^2}{6} = \frac{b(2b)^2}{6} = \frac{2b^3}{3}$$

$$\text{Wkt } M = Z \times \sigma_{\max}$$

$$111585.9375 \times 10^3 = 8 \times \frac{2b^3}{3}$$

$$b^3 = 20.9223 \times 10^6$$

$$b = 275.5 \text{ mm}$$

$$d = 2 \times 275.5 = 551 \text{ mm}$$

Problem 3.11: A rolled steel joist of I section has the dimensions as shown. This beam of I section carries a UDL of 40 kN/m run on a span of 10m. Calculate the max stress produced due to bending

Given

$$\text{UDL } w = 40 \text{ kN/m} = 40000 \text{ N/m}$$

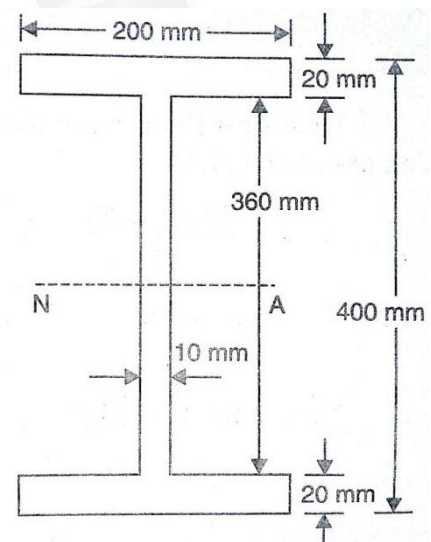
$$\text{Span } L = 10 \text{ m}$$

Solution:

Moment of inertia about the neutral axis

$$= 200 \times 400^3 / 12 - (200 - 10) \times 360^3 / 12$$

$$= 327946666 \text{ mm}^4$$



$$\begin{aligned}\text{Max BM. } M &= w \times L^2 / 8 = \frac{40000 \times 10^2}{8} = 500000 \text{ Nm} \\ &= 5 \times 10^8 \text{ N mm}\end{aligned}$$

Now using the relation $\frac{M}{I} = \frac{\sigma}{y}$

$$\therefore \sigma = \frac{M}{I} \times y$$

$$\begin{aligned}\sigma_{\max} &= \frac{M}{I} \times y_{\max} \\ &= 5 \times \frac{10^8}{327946666} \times 200 \\ &= \mathbf{304.92 \text{ N/mm}^2}\end{aligned}$$

Problem 3.12: An I section shown in figure is simply supported over a span of 12m. If the max permissible bending stress is 80 N/mm^2 . What concentrated load can be carried at a distance of 4m from one support?

Given

Bending stress $\sigma_{\max} = 80 \text{ N/mm}^2$

Load $W = 4\text{m}$ from support B in N

Solution:

Taking moments about A, we get

$$R_B \times 12 = W \times 8$$

$$\therefore R_B = \frac{8W}{12} = \frac{2}{3}W$$

and $R_A = W - R_B = W - \frac{2}{3}W = \frac{W}{3}$

B.M at C = $R_A \times 8 = \frac{8}{3}W \text{ Nm}$

But B.M at C is maximum

$$\therefore \text{Max B.M } M_{\max} = \frac{8}{3}W \text{ Nm}$$

$$= \frac{8}{3}W \times 1000 \text{ N mm}$$

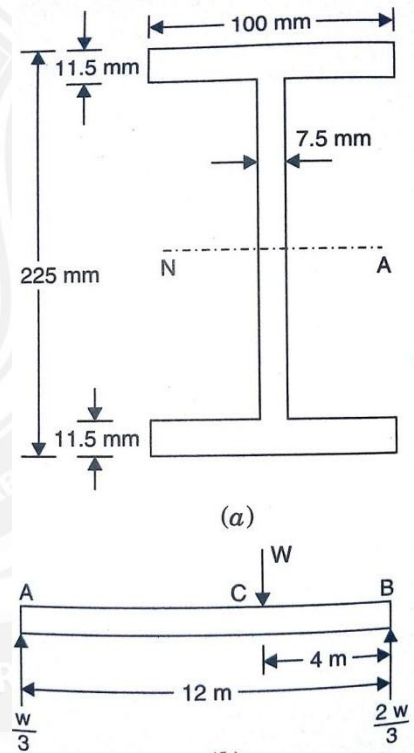
$$= \frac{8000}{3}W \text{ Nmm}$$

Now find the MOI of the given I section about the N.A

$$\begin{aligned}\therefore I &= \frac{100 \times 225^3}{12} - \frac{(100 - 7.5) \times (225 - 2 \times 11.5)^3}{12} \\ &= 3138647.45 \text{ mm}^4\end{aligned}$$

Now using the relation, $\frac{M}{I} = \frac{\sigma}{y}$

$$y_{\max} = \frac{225}{2} = 112.5 \text{ mm}$$



Now substituting the values, we get

$$\frac{\left(\frac{8000}{3}W\right)}{31386647.45} = \frac{80}{112.5}$$

$$W = 8369.77 \text{ N}$$

Problem 3.13: Two circular beams where one is solid of dia. D and other is a hollow of outer dia. D_0 and the inner dia. D_i are of same length, same material and of same weight. Find the ratio of section modulus of these circular beams.

Given

Dia. Of solid beam = D

Dia. Of hollow beam = D_0 and D_i

Let L be the length W be the weight and p be the density

Solution:

$$\begin{aligned} \text{Weight of solid beam} &= \rho \times g \times \text{area of section} \times L \\ &= \rho \times g \times L \times \pi/4 D^2 \end{aligned}$$

$$\begin{aligned} \text{Weight of hollow beam} &= \rho \times g \times \text{area of section} \times L \\ &= \rho \times g \times L \times \frac{\pi}{4} (D_0^2 - D_i^2) \end{aligned}$$

$$\begin{aligned} \text{But the weights are same } \rho \times g \times L \times \pi/4 D^2 &= \rho \times g \times L \times \frac{\pi}{4} (D_0^2 - D_i^2) \\ D^2 &= (D_0^2 - D_i^2) \end{aligned} \quad \dots\dots\dots(1)$$

$$\text{Now the section modulus of solid section } Z = \frac{\pi D^3}{32}$$

Section modulus of hollow section

$$\begin{aligned} Z_1 &= \frac{\pi (D_0^4 - D_i^4)}{32 D_0} \\ &= \frac{\pi (D_0^2 - D_i^2)(D_0^2 + D_i^2)}{32 D_0} \end{aligned}$$

$$\therefore \frac{\text{Section modulus of solid section}}{\text{Section Modulus of hollow section}}$$

$$\begin{aligned} &= \frac{\pi D^3}{32 D_0 (D_0 - D_i)(D_0 + D_i)} \\ &= \frac{D^3 \times D_0}{(D_0^2 - D_i^2)(D_0^2 + D_i^2)} = \frac{D \times D_0 \times D^2}{(D_0^2 - D_i^2)(D_0^2 + D_i^2)} \end{aligned}$$

Substitute eqn.1 in the above calculation, then

$$\begin{aligned} &= \frac{D \times D_0 \times (D_0^2 - D_i^2)}{(D_0^2 - D_i^2)(D_0^2 + D_i^2)} \\ &= \frac{D \times D_0}{D_0^2 + D_i^2} \end{aligned} \quad \dots\dots\dots(2)$$

Also from eqn.1

$$D_o^2 = D_o^2 - D_i^2 \quad \text{or} \quad D_i^2 = D_o^2 - D_o^2$$

Substituting the value D_i^2 in the section modulus ratio

$$\begin{aligned} \frac{\text{Section modulus of solid section}}{\text{Section Modulus of hollow section}} &= \frac{\frac{\pi D_o^3}{32}}{\frac{\pi (D_o^3 - D_i^3)}{32}} = \frac{D_o^3}{D_o^3 - D_i^3} \\ \text{or} \quad \frac{\text{Section modulus of hollow section}}{\text{Section Modulus of solid section}} &= \frac{D_o^3 - D_i^3}{D_o^3} = \frac{D_o^3 - (D_o^3 - D_o^3)}{D_o^3} \\ &= \frac{D_o^3}{D_o^3} = 1 \end{aligned}$$

Problem 3.14: A water main of 500mm internal dia and 20mm thick is running full. The water main is of cast iron and is supported at two points 10m apart. Find the max stress in the metal. The cast iron and water weigh 72000 N/m³ and 10000 N/m³ respectively.

Given:

Internal diameter $D_i = 500\text{mm} = 0.5\text{m}$

Thickness of pipe $t = 20\text{mm} = 0.02$

Outer diameter $= D_o = D_i + 2 \times t = 0.5 + 2 \times 0.02 = 0.54\text{m}$

Length of pipe $L = 10\text{m}$

Weight density of cast iron = 72000 N/m³

Weight density of water = 10000 N/m³

Solution:

$$\text{Internal area of pipe} = \frac{\pi D_i^2}{4} = \frac{\pi \times 0.5^2}{4} = 0.1960 \text{ m}^2$$

This is equal to the Area of water section

$$\therefore \text{Area of water section} = 0.1960 \text{ m}^2$$

area of pipe

$$= \frac{\pi D_o^2}{4} - \frac{\pi D_i^2}{4}$$

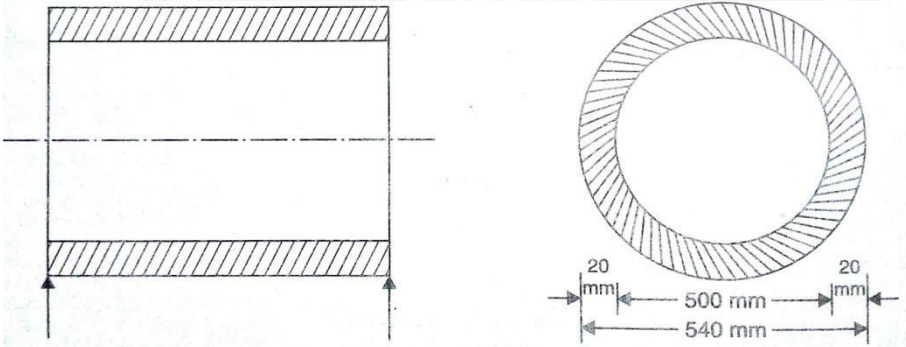


Diagram of a pipe section showing internal diameter 500 mm, outer diameter 540 mm, and thickness 20 mm.

Outer

$$= \frac{\pi D_o^2}{4} = \frac{\pi}{4}$$

Area of section

$$= \frac{\pi D_o^2}{4} - \frac{\pi D_i^2}{4}$$

$$= \frac{\pi (D_o^2 - D_i^2)}{4} = \frac{\pi (0.54^2 - 0.5^2)}{4} = 0.0327 \text{ m}^2$$

Moment of Inertia of pipe section about the neutral axis

$$I = \frac{\pi}{64} (D_o^4 - D_i^4) = \frac{\pi}{64} (0.54^4 - 0.5^4) = 1.105 \times 10^9 \text{ m}^4$$

Let us now find the weight of pipe and weight of water for one meter length,

Weight of pipe for one meter length

$$\begin{aligned} &= \text{weight density of cast iron} \times \text{volume of pipe} \\ &= 72000 \times \text{area of pipe section} \times \text{length} \\ &= 72000 \times 0.0327 \times 1 \quad (\because \text{length} = 1\text{m}) \\ &= 2354 \text{ N} \end{aligned}$$

Weight of water for one meter run

$$\begin{aligned} &= \text{wt. density of water} \times \text{volume of water} \\ &= 10000 \times \text{area of water section} \times \text{length} \\ &= 10000 \times 0.196 \times 1 = 1960 \text{ N} \end{aligned}$$

$$\therefore \text{Total weight on pipe for one meter run} = 2354 + 1960 = 4314 \text{ N}$$

Hence the above weight is the UDL on the pipe. the maximum bending moment due

to UDL is $\frac{w \times L^2}{8}$, where w = rate of UDL = 4314 N per meter length of beam.

$$\begin{aligned} \therefore \text{Max bending moment due to UDL } M &= \frac{w \times L^2}{8} \\ &= \frac{4314 \times 10^2}{8} = 53925 \text{ Nm} = 5.392 \times 10^7 \text{ Nmm} \end{aligned}$$

Now using

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\therefore \sigma = \frac{M}{I} \times y$$

to find the maximum stress it will act at y is maximum

$$y = \frac{D_o}{2} = \frac{540}{2} = 270 \text{ mm}$$

$$\therefore y_{\max} = 270 \text{ mm}$$

$$\therefore \sigma_{\max} = \frac{5.392 \times 10^7}{1.105 \times 10^9} \times 270 = 13.18 \text{ N/mm}^2$$

BENDING STRESS IN UNSYMMETRICAL SECTION

In case of symmetrical section, the neutral axis passes through the geometrical centre of the section. but in case of unsymmetrical section such as L, T section, the neutral axis does not pass through the geometrical centre of the section. Hence the value of y for the topmost layer or bottom layer of the section from neutral axis will not be same. For finding the bending stress in the beam, the bigger value of y is used. As the neutral axis passes through the

centre of gravity of the section, hence in unsymmetrical section, first the centre of gravity is calculated and followed to find other values.

Problem 3.15: A cast iron bracket subject to bending has the cross section of I form with unequal flanges. The dimensions of the sections are shown. Find the position of the neutral axis and M.O.I. of the section about the neutral axis. If the max bending moment on the section is 40 MN mm. Determine the max bending stress. What is the nature of the stress.

Given

Max. BM $M = 40 \text{ MN mm}$

Wkt, centroid about x axis $\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}$

Where

$$A_1 = \text{Area of bottom flange} = b_1 \times d_1 = 130 \times 50 = 6500 \text{ mm}^2$$

$$y_1 = \text{distance of C.G of } A_1 \text{ from bottom face} = \frac{50}{2} = 25 \text{ mm}$$

$$A_2 = \text{area of web} = b_2 \times d_2 = 50 \times 200 = 10000 \text{ mm}^2$$

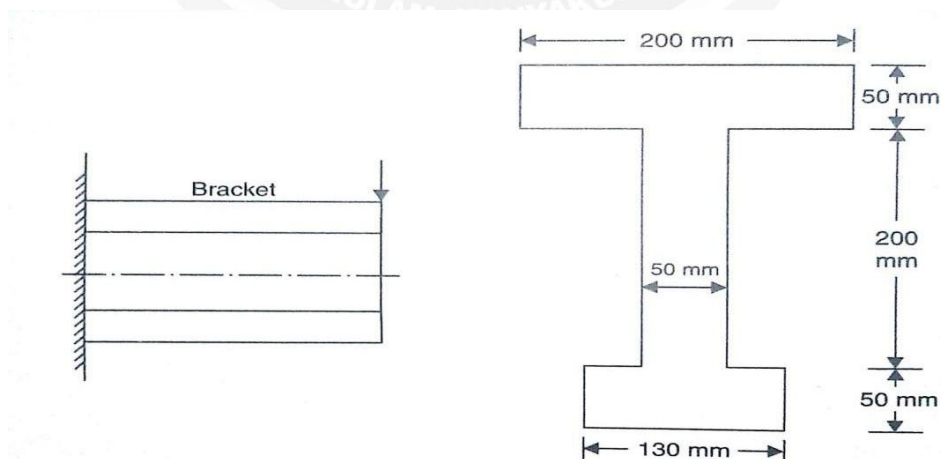
$$y_2 = \text{distance of C.G of } A_2 \text{ from bottom face} = 50 + \frac{200}{2} = 150 \text{ mm}$$

$$A_3 = \text{area of top flange} = b_3 \times d_3 = 200 \times 50 = 10000 \text{ mm}^2$$

$$y_3 = \text{dist of C.G of } A_3 \text{ from bottom face} = 50 + 200 + \frac{50}{2} = 275 \text{ mm}$$

Then,

$$\bar{y} = \frac{6500 \times 25 + 10000 \times 150 + 10000 \times 275}{6500 + 10000 + 10000} = 166.51 \text{ mm}$$



Hence the neutral axis is at a distance of 166.51 mm from the bottom face

M.O.I of the section about the neutral axis $I = I_1 + I_2 + I_3$

Where

$I_1 = \text{M.O.I of bottom flange about N.A}$

$$= I_{C.G} + A_1 \times (\text{distance of its C.G. from N.A.})^2$$

$$= \frac{b_1 \times d_1^3}{12} + A_1 (\bar{y}_1)^2$$

$$= \frac{130 \times 50^3}{12} + 6500 \times (166.51 - 25)^2 = 131517186.6 \text{ mm}^4$$

Similarly

$$I_2 = \frac{b_2 \times d_2^3}{12} + A_2 (\bar{y} - y_2)^2$$

$$= \frac{50 \times 200^3}{12} + 10000 \times (166.51 - 150)^2$$

$$= 3360913.43 \text{ mm}^4.$$

$$I_3 = \frac{b_3 \times d_3^3}{12} + A_3 (\bar{y} - y_3)^2$$

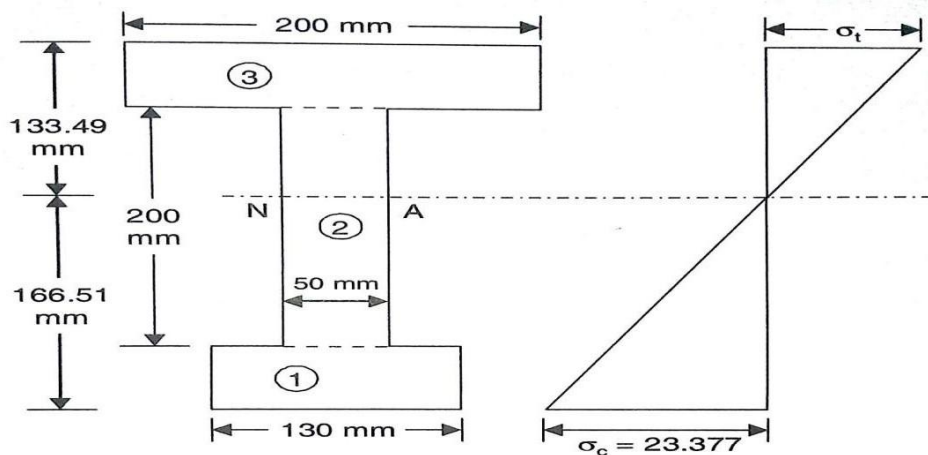
$$= \frac{200 \times 50^3}{12} + 10000 \times (166.51 - 275)^2$$

$$= 119784134.3 \text{ mm}^4.$$

∴

$$I = I_1 + I_2 + I_3 = 131517186.6 + 3360913.43 + 119784134.3$$

$$= 284907234.9 \text{ mm}^4$$



Now distance of C.G. from the upper top fibre

$$= 300 - \bar{y} = 300 - 166.51 = 133.49 \text{ mm}$$

Distance of C.G. from the bottom fibre = 166.51 mm

Hence we shall take the value of $y = 166.51 \text{ mm}$ for maximum bending stress.

Now using bending equation

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{M}{I} \times y = \frac{40 \times 10^6}{284907234.9} \times 166.51 = 23.377 \text{ N/mm}^2$$

∴ Max bending stress = **23.377 N/mm²**

Problem 3.16: A cast iron beam is of I section. The beam is simply supported on a span of 5m. If the tensile stress is not to exceed 20 N/mm^2 . Find the safe uniformly load which the beam can carry. Find also the max compressive stress.

Given

Length $L = 5\text{m}$

Max tensile stress $\sigma_t = 20\text{N/mm}^2$

Solution:

Wkt, centroid about x axis $\bar{y} = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A_1 + A_2 + A_3}$

Where

A_1 = Area of bottom flange

$$= b_1 \times d_1 = 160 \times 40$$

$$= 6400\text{mm}^2$$

y_1 = distance of C.G of A_1 from bottom face

$$= \frac{40}{2} = 20\text{mm}$$

A_2 = area of web $= b_2 \times d_2 = 20 \times 200 = 4000\text{mm}^2$

y_2 = distance of C.G of A_2 from bottom face $= 40 + \frac{200}{2} = 140\text{mm}$

A_3 = area of top flange $= b_3 \times d_3 = 80 \times 20 = 1600\text{mm}^2$

y_3 = dist of C.G of A_3 from bottom face $= 40 + 200 + \frac{20}{2} = 250\text{mm}$

Then,

$$\bar{y} = \frac{6400 \times 20 + 4000 \times 140 + 1600 \times 250}{6400 + 4000 + 1600} = 90.66\text{mm}$$

Hence the neutral axis is at a distance of 166.51mm from the bottom face or $260 - 90.66 = 169.34\text{mm}$ from the top face.

M.O.I of the section about the neutral axis $I = I_1 + I_2 + I_3$

Where

I_1 = M.O.I of bottom flange about N.A

$$= I_{C.G} + A_1 \times (\text{distance of its C.G. from N.A.})^2$$

$$= \frac{b_1 \times d_1^3}{12} + A_1 (\bar{y} - y_1)^2$$

$$= \frac{160 \times 40^3}{12} + 6400 \times (90.66 - 20)^2 = 32807481.17\text{mm}^4$$

Similarly

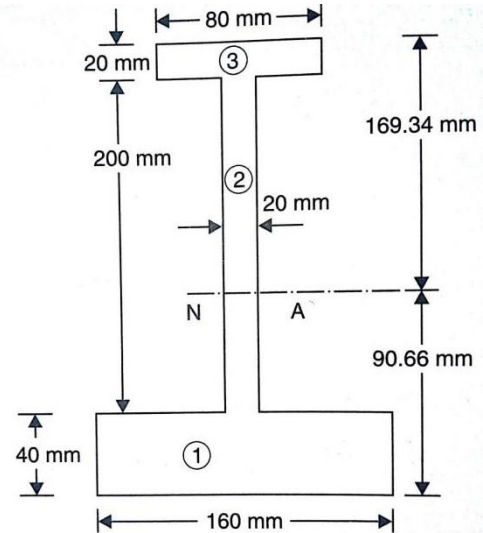
$$I_2 = \frac{b_2 \times d_2^3}{12} + A_2 (\bar{y} - y_2)^2$$

$$= \frac{20 \times 200^3}{12} + 4000 \times (90.66 - 140)^2$$

$$= 23071075.73\text{mm}^4$$

$$I_3 = \frac{b_3 \times d_3^3}{12} + A_3 (\bar{y} - y_3)^2$$

$$= \frac{80 \times 20^3}{12} + 1600 \times (90.66 - 250)^2$$



$$= 40676110.29 \text{ mm}^4$$

$$\therefore I = I_1 + I_2 + I_3 = 32807481.17 + 23071075.73 + 40676110.29$$

$$= 96554667.21 \text{ mm}^4$$

For a simply supported beam the tensile stress will be at the extreme bottom and the compressive stress will be at the extreme top fibre

$$\text{Here max. tensile stress} = 20 \text{ N/mm}^2$$

$$\text{Hence for max tensile stress } y = 90.66 \text{ mm}$$

$$\text{Using } \frac{M}{I} = \frac{\sigma}{y}$$

$$M = \frac{\sigma}{y} \times I$$

$$= \frac{20}{90.66} \times 96554667.21$$

$$= 21300389.85 \text{ Nmm}$$

Let w = uniformly distributed load in N/m on the simply supported beam

Max bending moment is at the centre and equal to $wL^2/8$

$$\therefore M = w \times 5^2 / 8 \text{ Nm} = w \times 25 \times 1000 / 8 \text{ N mm} = 3125w \text{ N mm}$$

Equating the values of M

$$3125w = 21300389.85$$

$$w = \mathbf{6816.125 \text{ N/m}}$$

Max compressive stress

Distance of extreme top fibre from N.A

$$y_c = 169.34 \text{ mm}$$

$$M = 21300389.85$$

$$I = 96554667.21$$

$$\text{Let } \sigma_c = \text{max compressive stress}$$

$$\text{Using the relation } \frac{M}{I} = \frac{\sigma}{y}$$

$$\therefore \sigma = \frac{M}{I} \times y = \frac{21300389.85}{96554667.21} \times 169.34$$

$$\sigma_c = \mathbf{37.357 \text{ N/mm}^2}$$

Problem 3.17: A cast iron beam is of T section as shown. The beam is simply supported on a span of 8m. The beam carries a uniformly distributed load of 1.5kN/m length on the entire span. Determine the max tensile and max compressive stress.

Given

Length $L = 8\text{m}$

U.D.L $w = 1.5\text{KN/m}$

Let $y =$ distance of C.G. of section from the bottom

$$Y = (A_1 F_1 + A_2 F_2) / (A_1 + A_2)$$

$$= (100 \times 20) \times (100 \times 20) \times (80 + (20/2)) + 80 \times 20 \times 80/2 / (100 \times 20 + 80 \times 20)$$

$$= 67.77\text{mm}$$

N.A lies at the distance of 67.77mm from the bottom face or $100 - 67.77 = 32.23\text{mm}$ from the top face

Moment of inertia $I = I_1 + I_2$

Where $I_1 =$ M.O.I of top flange about N.A

$$= \frac{100 \times 20^3}{12} + (100 \times 20) \times (32.23 - 10)^2$$

$$= 1055012.5\text{mm}^4$$

$I_2 =$ M.O.I of web

$$= 20 \times 80^3 / 12 + (80 \times 20) \times (67.77 - 40)^2$$

$$= 2087209.9\text{mm}^4$$

$$I = I_1 + I_2 = 1055012.5 + 2087209.9\text{mm}^4$$

$$= 3142222.4\text{mm}^4$$

For simply supported beam the tensile stress will be at the extreme bottom and the compressive stress will be at the extreme top

$$\text{Max B.M } M = w \times L^2 / 8 = 1500 \times 8^2 / 8 = 12000\text{ Nm}$$

$$= 12000000\text{ Nmm}$$

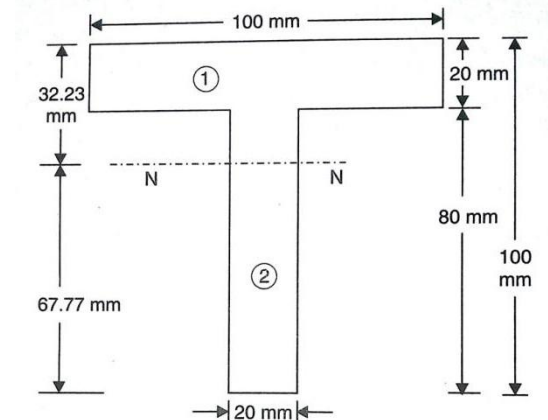
Now using relation $\frac{M}{I} = \frac{\sigma}{y} \quad \therefore \sigma = \frac{M}{I} \times y$

1. For max tensile stress $y = 67.77\text{mm}$

$$\sigma = \frac{12000000}{3142222.4} \times 67.77 = \mathbf{258.81\text{ N/mm}^2}$$

2. for max compressive stress $y = 32.33\text{mm}$

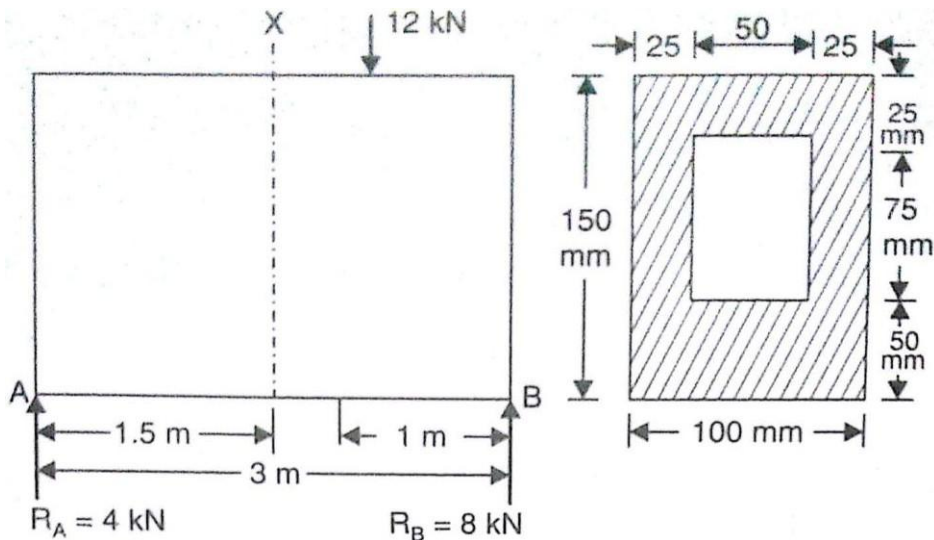
$$\sigma = \frac{12000000}{3142222.4} \times 32.23 = \mathbf{123.08\text{ N/mm}^2}$$



Problem 3.18: A simply supported beam of length 3m carries a point load of 12k N at a distance of 2m from left support. The cross section of the beam is shown. Determine the max tensile and compressive stress at X-X

Given

Point load $w = 12 \text{ kN} = 1200 \text{ N}$



Solution:

First find the B.M at X-X.

Taking moments about A

$$R_B \times 3 = 12 \times 2$$

$$R_B = 8 \text{ kN}$$

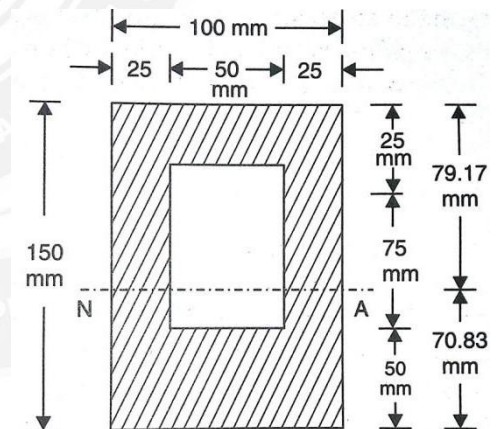
$$R_A = W - R_B$$

$$= 4 \text{ kN}$$

$$\text{B.M. at X-X} = R_A \times 1.5 = 4 \times 1.5 = 6 \text{ kNm}$$

$$= 6000 \times 1000 \text{ N mm}$$

$$M = 6000,000 \text{ N mm}$$



Let y = distance of C.G. of the section from the bottom edge

$$= (A_1 F_1 - A_2 F_2) / (A_1 - A_2)$$

$$= (150 \times 100) \times 75 - (75 \times 50) \times (50 + (75/2)) / (150 \times 100 - 75 \times 50)$$

$$= 70.83 \text{ mm}$$

Hence N.A will lie at a distance of 70.83mm from the bottom edge or $150 - 70.83 = 79.17 \text{ mm}$ from the top edge

$$\text{M.O.I of section } I = I_1 - I_2$$

Where I_1 = M.O.I of outer rectangle about N.A

$$\begin{aligned}
 &= \text{M.O.I of rectangle } 100 \times 150 \text{ about its C.G} + A_1 \\
 &= 100 \times 150^3 / 12 + 100 \times 150 \times (75 - 70.83)^2 \\
 &= 28385833.5 \text{ mm}^4
 \end{aligned}$$

$$\begin{aligned}
 I_2 &= \text{M.O.I of cut out part about N.A} \\
 &= 50 \times 75^3 / 12 + 50 \times 75 \times (50 + 75/2 - 70.83)^2 \\
 &= 2799895.875 \text{ mm}^4
 \end{aligned}$$

$$I = 28385833.5 - 2799895.875 = 25585937.63 \text{ mm}^4$$

The bottom edge of the section will be subjected to tensile stress whereas the top edge will be subjected to compressive stress. the top edge is at 79.17 mm from N.A whereas bottom edge is 70.83 mm from N.A

Now using relation $\frac{M}{I} = \frac{\sigma}{y} \quad \therefore \sigma = \frac{M \times y}{I}$

For max tensile stress $y = 70.83 \text{ mm}$

$$\text{Max tensile stress } \sigma = \frac{6000000}{25585937.63} \times 70.83 = \mathbf{16.60 \text{ N/mm}^2}$$

For max compressive stress $y = 79.17 \text{ mm}$

$$\sigma = \frac{6000000}{25585937.63} \times 79.17 = \mathbf{18.56 \text{ N/mm}^2}$$

STRENGTH OF A SECTION

The strength of a section means the moment of resistance offered by the section and moment of resistance is given by

$$M = Z \times \sigma$$

Where M = moment of resistance

σ = bending stress

Z = section modulus

Problem 3.19: Three beams have the same length same allowable bending stress and the same bending moment. The cross section of the beam are a square rectangle with depth twice the width and a circle. Find the radius of weights of the circular and the rectangular beams with respect to square beams.

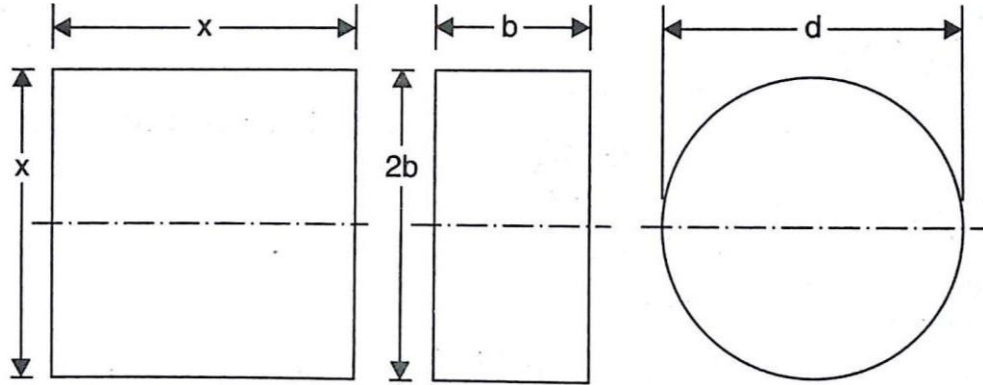
Given

Let x = side of square beam

b = width of rectangular beam

$2b$ = depth of rectangular beam

D = diameter of circular section



Solution:

$$= \frac{x^3}{6}$$

$$\text{Section modulus of rectangular beam} = \frac{\frac{b \times (2b)^3}{12}}{\frac{2b}{2}} = \frac{2}{3} b^3$$

$$\text{Section modulus of a circular beam} = \frac{\frac{\pi d^4}{64}}{\frac{d}{2}} = \frac{\pi d^3}{32}$$

Equating the section modulus of square beam with that of rectangular beam

$$\begin{aligned} \frac{x^3}{6} &= \frac{2}{3} b^3 \\ b^3 &= 0.25x^3 \\ b &= 0.63x \end{aligned}$$

Equating the section modulus of square beam with that of a circular beam

$$\begin{aligned} \frac{x^3}{6} &= \frac{\pi d^3}{32} \\ d^3 &= \frac{32x^3}{6\pi} \\ d &= 1.1927x \end{aligned}$$

Weight of beams are proportional to their cross sectional areas. Hence

$$\frac{\text{Weight of rectangular beam}}{\text{weight of square beam}} = \frac{\text{Area of rectangular beam}}{\text{Area of square beam}}$$

$$\frac{b \times 2b}{x \times x} = \frac{0.63x \times 2 \times 0.63x}{x \times x} = \mathbf{0.7938}$$

$$\frac{\text{Weight of rectangular beam}}{\text{weight of square beam}} = \frac{\text{Area of rectangular beam}}{\text{Area of square beam}}$$

$$\frac{\pi d^2}{4x^2} = \frac{\pi d^2}{4x^2} \quad (\because d = 1.1927 x)$$

$$= \frac{\pi (1.1927 x)^2}{4x^2} = \mathbf{1.1172}$$

