5.2 Euler's method - Modified Euler's method

Euler's method

$$y_{n+1} = y_n + h\, f(x_n, y_n), n = 0,1,2,3, \ldots.$$

Modified Euler's method

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}h\, f(x_n, y_n)\right)\right]$$

1. Using Euler's method find $y(0.2) and\ y(0.4)$ from $\frac{dy}{dx} = x + y,$
$$y(0) = 1\ with\ h = 0.2$$

Solution

Given, $f(x, y) = x + y,\ \ x_0 = 0, y_0 = 1, h = 0.2, x_1 = 0.2, x_2 = 0.4$

Euler's algorithm is,

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$y_1 = y(0.2) = 1 + (0.2)(x_0 + y_0)$$

$$= 1 + (0.2)(0 + 1)2$$

$$y_1 = y(0.2) = 1.2$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$y_2 = y(0.4) = 1.2 + (0.2)(x_1 + y_1)$$

$$= 1.2 + (0.2)(0.2 + 1.2)$$

$$= 1.2 + 0.28$$

$$y_2 = y(0.2) = 1.48$$

2. Using Euler's method find the solution of the initial value problem $\frac{dy}{dx} = \log(x + y),\ y(0) = 2$by assuming $h = 0.2$

Solution

Given, $f(x, y) = \log(x + y),\ \ x_0 = 0, y_0 = 2, h = 0.2, x_1 = 0.2$

Euler's algorithm is,

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$= y_0 + h\,[\log(x_0 + y_0)]$$

$$y_1 = y(0.2) = y_0 + h\,[\log(x_0 + y_0)]$$

$$= y(0.2) = 2 + (0.2)\,[\log(0 + 2)]$$

$$= 2 + (0.2)[\log(2)]$$

$$= 2 + (0.2)(0.3010)$$

$$= 2.0602$$

$$= y(0.2) = 2.0602$$

3. $Using\ Euler\ Method\ solve\ y' = x + y + xy\,, y(0) = 1,\ compute\ y\ at$ $x = 0.05$

$solution$:

$$Given\ y' = f(x, y) = x + y + xy\ \&$$

$$x_0 = 0\ and\ y_0 = 1,\ \ x_1 = 0.05\ and\ y_1 = ?$$

$$h = x_1 - x_0 = 0.05 - 0 = 0.05$$
$$y_1 = y_0 + hf(x_0, y_0)$$
$$y_1 = 1 + 0.05\,f(0,1)$$
$$= 1 + 0.05\,[0 + 1 + (0)(1)]$$

$$= 1 + 0.05(1)$$

$$y_1 = y(0.05) = 1.05$$

4. Using modified Euler's method, compute y (0.1) with h = 0.1 from $y' = y - \frac{2x}{y}, y(0) = 1$

Solution :

Given $f(x, y) = y - \frac{2x}{y},\ \ x_0 = 0, y_0 = 1, h = 0.1, x_1 = 0.1$

By modified Euler's method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}h\,f(x_n, y_n)\right)\right],$$

$$y_1 = y(0.1) = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}h\,f(x_0, y_0)\right)\right],$$

$$= y(0.1) = 1 + 0.1\left[f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\left[y_0 - \frac{2x_0}{y_0}\right]\right)\right]$$

$$= 1 + 0.1\,[f(0.05, 1 + (0.05)[1 - 0])]$$

$$= 1 + 0.1\,[f(0.05, 1.05)]$$

$$= 1 + 0.1\left[1.05 - \frac{2(0.05)}{1.05}\right]$$

$$= 1 + 0.1[1.05 - 0.0952]$$

$$= 1 + 0.1[0.9548]$$

$$= 1 + 0.09548$$

$$= 1.09548$$

5. Consider the initial value problem $\frac{dy}{dx} = y - x^2 + 1, y(0) = 0.5,$ using the modified Euler's method, find $y(0.2)$

Solution :

Given $f(x, y) = y - x^2 + 1,\ \ x_0 = 0, y_0 = 0.5, h = 0.2, x_1 = 0.2$

By modified Euler's method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}h\,f(x_n, y_n)\right)\right]$$

$$y_1 = y(0.2) = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}h\,f(x_0, y_0)\right)\right]$$

$$= y(0.2) = 0.5 + (0.2)\left[f\left(0 + \frac{0.2}{2}, 0.5 + \frac{0.2}{2}\ f(0, 0.5)\right)\right]$$

$$= 0.5 + 0.2\ [f(0.1, 0.5 + (0.1)[0.5 - 0 + 1])]$$

$$= 0.5 + 0.2\ [f(0.1, 0.5 + (0.1)(1.5))]$$

$$= 0.5 + 0.2\ [f(0.1, 0.5 + 0.15)]$$

$$= 0.5 + 0.2\ [f(0.1, 0.65)]$$

$$= 0.5 + 0.2\ [0.65 - 0.1^2 + 1]$$

$$= 0.5 + 0.2\ [0.65 - 0.01 + 1]$$

$$= 0.5 + 0.2\ [1.64]$$

$$= 05 + 0.328 = 0.828$$

6. $Solve\ y' = 1 - y\ , y(0) = 0\ \ byModified\ Euler\ Method$
   $find\ y(0.1)$

$Solution$:

$Given\ y' = f(x, y) = 1 - y$

$$x_0 = 0 \quad and \quad y_0 = 0$$
$$x_1 = 0.1 \quad and \quad y_1 = ?$$
$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$
$$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right)\right]$$
$$y_1 = 0 + 0.1\left[f\left(0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}f(0,0)\right)\right]$$
$$= 0.1\left[f\left(0.05, \frac{0.1}{2}(1-0)\right)\right]$$
$$= 0.1\left[f\left(0.05, 0.05(1)\right)\right]$$
$$= 0.1[f(0.05, 0.05)]$$
$$= 0.1[1 - 0.05]$$
$$= 0.1[0.95]$$
$$y_1 = 0.095$$