ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MATHEMATICS

UNIT II – FOURIER SERIES

2.5 EVEN & ODD FUNCTION - HALF RANGE FOURIER SERIES

A function $y = f(x)$ is an:

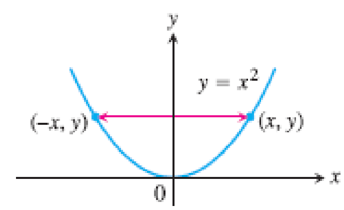
Evenfunction of x if $f(-x) = f(x)$,

Odd function of x if $f(-x) = -f(x)$,

for every $x$in the function's domain.

The graphs of even and odd functions have characteristic symmetry properties.

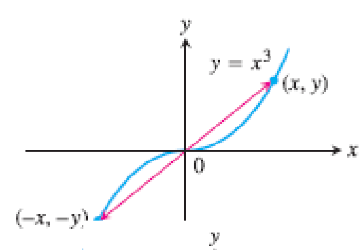
- The graph of an **even function** is **symmetric about the y-axis**.
- The graph of an **odd function** is **symmetric about the origin**.



Problem 1:

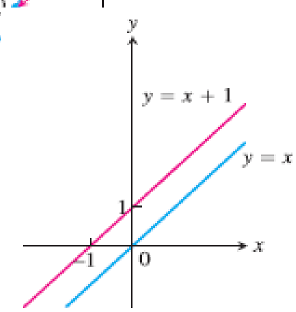
$f(x) = x^2$is an even function, since$(-x)^2 = x^2$ for all x, (symmetry about y-axis).

Problem 2:



$f(x) = x^3$is an odd function, since$(-x)^3 = -x^3$ for all x, (symmetry about the origin).

Problem 3:



$f(x) = x$is an odd function, since$(-x) = -x$ for all x, (symmetry about the origin).

Problem 4:

$f(x) = x + 1$ is neither even nor odd:

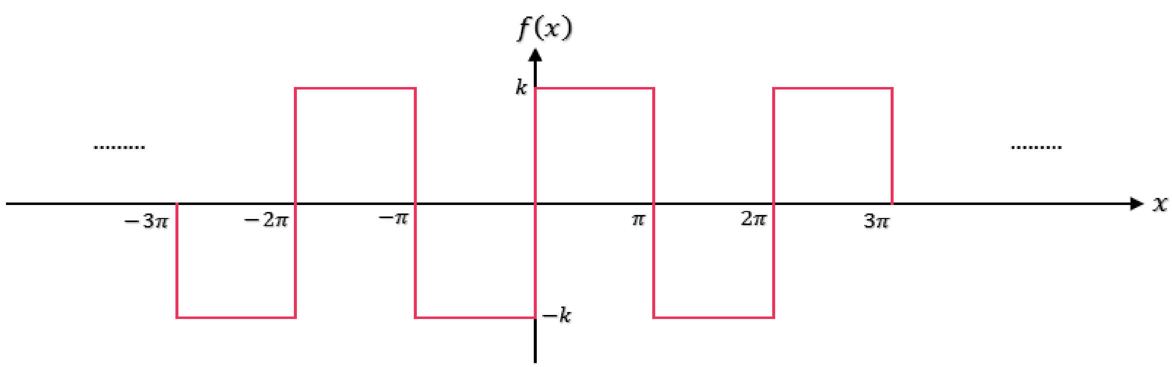
Not even function since, $f(-x) \neq f(x)$for all $x \neq 0$where $f(-x) = (-x) + 1$, but $f(x) = x + 1$.

Not odd function since, $f(-x) \neq -f(x)$ where $f(-x) = -x + 1$, but $-f(x) = -x - 1$.

Problem 5: Find the Fourier series corresponding to the following periodic function:

$$f(x) = \begin{cases} -k, & for \; -\pi < x \leq 0 \\ k, & for \quad 0 < x \leq \pi \end{cases}$$

where$f(x) = f(x + 2\pi)$



Solution:

The period $T = 2L = 2\pi \Rightarrow \; L = \pi$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-k)dx + \frac{1}{\pi}\int_{0}^{\pi}(k)dx$$

$$= \frac{-k}{\pi}x]_{-\pi}^{0} + \frac{k}{\pi}x]_{0}^{\pi}$$

$$= \frac{-k}{\pi}(0+\pi) + \frac{k}{\pi}(\pi - 0) = -k + k = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-k)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(k)\cos nx\,dx$$

$$= \frac{-k}{n\pi}\sin nx\Big]_{-\pi}^{0} + \frac{k}{n\pi}\sin nx\Big]_{0}^{\pi}$$

$$= \frac{-k}{n\pi}[\sin 0 - \sin(-n\pi)] + \frac{k}{n\pi}[\sin \pi n - \sin 0] = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{\pi}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-k)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(k)\sin nx\,dx$$

$$= \frac{k}{n\pi}\cos nx\Big]_{-\pi}^{0} - \frac{k}{n\pi}\cos nx\Big]_{0}^{\pi}$$

$$= \frac{k}{n\pi}[\cos 0 - \cos(-n\pi)] - \frac{k}{n\pi}(\cos n\pi - \cos 0)$$

$$= \frac{k}{n\pi}[1 - \cos(n\pi)] - \frac{k}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2k}{n\pi}[1 - \cos(n\pi)]$$

$$= \frac{2k}{n\pi}\left(1 - (-1)^n\right) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4k}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

we can write the even- indexed and odd-indexed coefficients separately as $b_{(2n)} = 0$, for $k = 1,2,\cdots$ and $b_{(2n-1)} = \frac{4k}{(2n-1)\pi}$, for $k = 1,2,\cdots$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

$$= \sum_{n=1}^{\infty} b_{(2n-1)} \sin\frac{(2n-1)\pi x}{\pi}$$

$$= \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi} \sin(2n-1)x$$

$$= \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \cdots$$

Half Range Fourier series

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$, which is half of the interval $(-L, L)$, and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, $(-L, 0)$.

For half range Fourier sine series (odd functions):

$$\boxed{a_0 = 0}$$

$$\boxed{a_n = 0}$$

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad for\ 0 \le x \le L}$$

For half range Fourier cosine series (even functions):

$$\boxed{a_0 = \frac{2}{L}\int_0^L f(x)\,dx}$$

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}$$

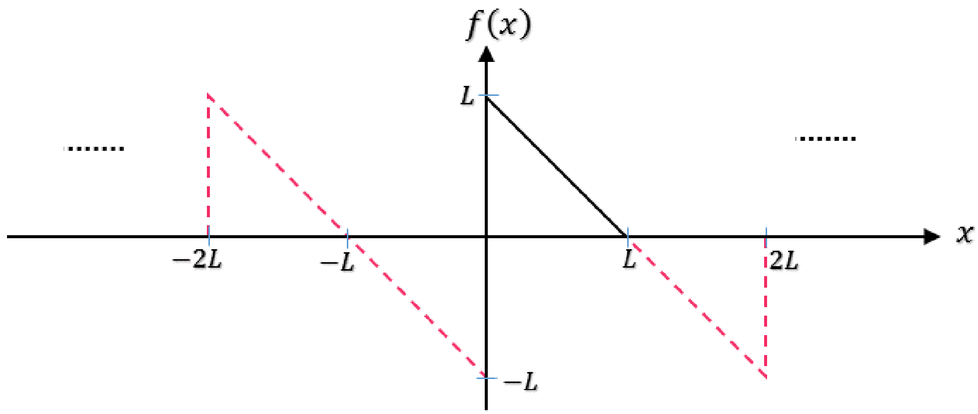
$$\boxed{b_n = 0}$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad for\ 0 \le x \le L}$$

Problem 6: Determine the half range Fourier sine series corresponding to: $f(x) = L - x,\ for\ 0 < x < L.$

Solution:

A Fourier series consisting of sine terms alone is obtained only for an odd function. Hence, we extend the definition of $f(x)$ so that it becomes odd.



Taking the period $T = 2L$

Since the function is odd then: $a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L (L-x)\left(\sin\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[(L-x)\left(\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2}\right)\right]_0^L$$

$$= \frac{2}{L}\left[\frac{L}{n\pi}(x-L)\left(\cos\left(\frac{n\pi x}{L}\right)\right) - \frac{L^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi x}{L}\right)\right)\right]_0^L$$

$$= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left[n\pi(x-L)\left(\cos\left(\frac{n\pi x}{L}\right)\right) - L\sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \left(\frac{2}{n^2\pi^2}\right)\left[0 - n\pi(0-L)(\cos 0) - L\sin\left(\frac{n\pi L}{L}\right) + L\sin 0\right]$$

$$= \left(\frac{2}{n^2\pi^2}\right)[n\pi L] = \frac{2L}{n\pi}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad for\ 0 \le x \le L$$

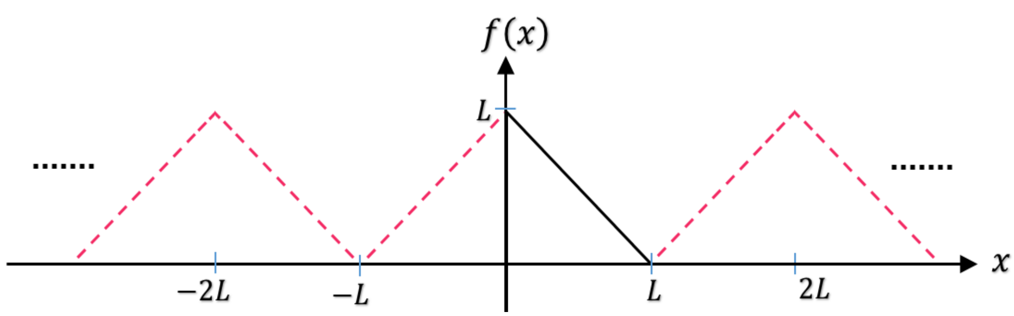
$$= \sum_{n=1}^{\infty} \left(\frac{2L}{n\pi}\right) \sin\frac{n\pi x}{L}$$

$$= \left(\frac{2L}{\pi}\right) \sin\frac{\pi x}{L} + \left(\frac{2L}{2\pi}\right) \sin\frac{2\pi x}{L} + \left(\frac{2L}{3\pi}\right) \sin\frac{3\pi x}{L} + \cdots$$

<u>Problem 7:</u> Determine the half range Fourier cosine series corresponding to: $f(x) = L - x,\ for\ 0 < x < L.$

<u>Solution:</u>

A Fourier series consisting of cosine terms alone is obtained only for an even function. Hence, we extend the definition of $f(x)$ so that it becomes even.



Taking the period $T = 2L$

Since the function is even then: $b_0 = 0$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{L}\int_0^L (L-x)\,dx = \frac{2}{L}\left[L\,x - \frac{x^2}{2}\right]_0^L = \frac{2}{L}\left(L^2 - \frac{L^2}{2}\right) = \frac{2}{L}\left(\frac{L^2}{2}\right)$$

$$= L$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L (L-x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[(L-x)\left(\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)}\right) - (-1)\left(\frac{-\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2}\right)\right]_0^L$$

$$= \frac{2}{L}\left[\frac{L}{n\pi}(L-x)\left(\sin\left(\frac{n\pi x}{L}\right)\right) - \frac{L^2}{n^2\pi^2}\left(\cos\left(\frac{n\pi x}{L}\right)\right)\right]_0^L$$

$$= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)[0 - n\pi(L-0)\sin(0) - L\cos(n\pi) + L\cos(0)]$$

$$= \left(\frac{2}{n^2\pi^2}\right)[-L\cos(n\pi) + L]$$

$$= \left(\frac{2L}{n^2\pi^2}\right)[1-(-1)^n] = \begin{cases} 0\,, & \text{if } n \text{ is even} \\ \frac{4L}{n^2\pi^2}\,, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad for\ 0 \le x \le L$$

$$= \frac{L}{2} + \sum_{n=1}^{\infty} a_{(2n-1)} \cos\frac{(2n-1)\pi x}{L}$$

$$= \frac{L}{2} + \sum_{n=1}^{\infty} \left(\frac{4L}{(2n-1)^2\pi^2}\right) \cos\frac{(2n-1)\pi x}{L}$$

$$= \frac{L}{2} + \left(\frac{4L}{(1)^2\pi^2}\right)\cos\frac{\pi x}{L} + \left(\frac{4L}{(3)^2\pi^2}\right)\cos\frac{3\pi x}{L} + \left(\frac{4L}{(5)^2\pi^2}\right)\cos\frac{5\pi x}{L} + \cdots$$

$$= \frac{L}{2} + \left(\frac{4L}{\pi^2}\right)\cos\frac{\pi x}{L} + \left(\frac{4L}{9\pi^2}\right)\cos\frac{3\pi x}{L} + \left(\frac{4L}{25\pi^2}\right)\cos\frac{5\pi x}{L} + \cdots$$