4.5 *Trapezoidal rule* and ***Simpson's*** $\frac{1}{3}$ ***rule***

Trapezoidal rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \ldots\ldots y_{n-1})]$$

1. $Evaluate \int_0^1 \frac{1}{1+x^2}\, dx \;\; with\ h = \frac{1}{6} \;\; by\ Trapezoidal\ Rule$

$Solution:\quad Let\ f(x) = \frac{1}{1+x^2} \quad and\ h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = f(x)$	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$

$$f(x) = \frac{1}{1+x^2}$$

$$f(0) = \frac{1}{1+0^2} = \frac{1}{1+0} = 1$$

$$f\left(\frac{1}{6}\right) = \frac{1}{1+\frac{1}{36}} = \frac{1}{\frac{36+1}{36}} = \frac{36}{37}$$

$$f\left(\frac{2}{6}\right) = \frac{1}{1+\frac{4}{36}} = \frac{1}{\frac{36+4}{36}} = \frac{36}{40} = \frac{9}{10}$$

By Trapezoidal Rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \ldots\ldots y_{n-1})]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/6}{2}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{36}{37}+\frac{9}{10}+\frac{4}{5}+\frac{9}{13}+\frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right)+2(3.9554)\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right)+7.9108\right]$$

$$= 0.7842$$

2. *Evaluate* $\int_1^2 \frac{1}{1+x^2}\,dx$ *with four sub intervals by Trapezoidal Rule*

Solution:

$$Let\ y = f(x) = \frac{1}{1+x^2} \quad and\ h = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

x	1	1.25	1.5	1.75	2
$y = f(x)$	0.5	0.3902	0.3077	0.2462	0.2

$$f(x) = \frac{1}{1+x^2}$$

$$f(1) = \frac{1}{1+1^2} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$f(1.25) = \frac{1}{1+(1.25)^2} = 0.3902$$

$$f(1.5) = \frac{1}{1+(1.5)^2} = 0.3077$$

Trapezoidal Rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \ldots\ldots y_{n-1})]$$

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{0.25}{2}[(0.5 + 0.2) + 2(0.3902 + 0.3077 + 0.2462)]$$

$$= 0.125[(0.7) + 2(0.9441)]$$

$$= 0.125[2.5882] = 0.3235$$

Simpson's $\frac{1}{3}$ rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \cdots.) \quad + 4(y_1 + y_3 + y_5 \ldots.)]$$

1. *Evaluate* $\int_0^1 \frac{1}{1+x^2}\, dx$ *with* $h = \frac{1}{6}$ *by Simpson's* $\frac{1}{3}$ *rule*

Solution :

$$Let\ f(x) = \frac{1}{1+x^2} \quad and\ h = \frac{1}{6}$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$

$\mathrm{f}(x) = \frac{1}{1+x^2}$

$$f(0) = \frac{1}{1+x^2} = \frac{1}{1+0} = 1$$

$$f\left(\frac{1}{6}\right) = \frac{1}{1+\frac{1}{36}} = \frac{1}{\frac{36+1}{36}} = \frac{36}{37}$$

$$f\left(\frac{2}{6}\right) = \frac{1}{1+\frac{4}{36}} = \frac{1}{\frac{36+4}{36}} = \frac{36}{40} = \frac{9}{10}$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$By\ Simpson's\ \frac{1}{3}\ rule$

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \cdots.) \quad + 4(y_1 + y_3 + y_5 \ldots.)]$$

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{1/6}{2}\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{9}{10}+\frac{9}{13}\right) + 4\left(\frac{36}{37}+\frac{4}{5}+\frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right) + 2(1.5923) + 4(2.3632)\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right) + 3.1846 + 9.4528\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right) + 12.6374\right]$$

$$= 1.1781$$

2. $Evaluate \int_{1}^{2} \frac{1}{1+x^2}\, dx$ $with\ Four\ sub\ interval\ by\ Simpson's\ \frac{1}{3}\ rule$

$Solution:$

$$Let\ f(x) = \frac{1}{1+x^2} \quad and\ h = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

x	1	1.25	1.5	1.75	1
y	0.5	0.3902	0.3077	0.2462	0.2
	y_0	y_1	y_2	y_3	y_4

$$f(x) = \frac{1}{1+x^2}$$

$$f(1) = \frac{1}{1+x^2} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$f(1.25) = \frac{1}{1+(1.25)^2} = 0.3902$$

$$f(1.5) = \frac{1}{1+(1.5)^2} = 0.3077$$

$$Simpson's\ \frac{1}{3}\ rule$$

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \cdots.) \quad + 4(y_1 + y_3 + y_5 \ldots.)]$$

$$\int_{0}^{1} \frac{1}{1+x^2}\, dx = \frac{0.25}{2}\big[\big((0.5+0.2)\big) + 2(0.3077) + 4(0.3902 + 0.2462)\big]$$

$$= 0.125[(0.7) + 0.6154 + 2.5456]$$

$$= 0.125[3.861]$$

$$= 0.4826$$

Trapezoidal rule for Double Integral

$$I = \frac{hk}{4}[(Sum\ of\ \ four\ corners) + 2(Sum\ of\ nodes\ on\ boundary) + 4(Sum\ of\ interior\ nodes)]$$

Simpson's $\frac{1}{3}$ rule forDouble Integral

Simpson's 1/3 rule $= \frac{hk}{9}[(Sum\ of\ the\ corner\ of\ the\ boundary)$

$$+2(sum\ of\ the\ odd\ nodes\ of\ the\ boundary) + 4(sum\ of\ the\ even\ nodes\ of\ the\ boundary) + 4(sum\ of\ the\ odd\ nodes\ of\ the\ odd\ rows) + 8(sum\ of\ the\ even\ nodes\ of\ the\ odd\ rows) + 8(sum\ of\ the\ odd\ nodes\ of\ the\ even\ rows) + 16(sum\ of\ the\ even\ nodes\ of\ the\ even\ rows)$$

1. $Evaluate \int_1^2 \int_3^4 \frac{1}{(x+y)^2}\ dxdy\ \ with\ h = k = 0.5\ by\ \boldsymbol{T}rapezoidal$

<u>$and\ Simpson's\ rule$</u>

$Solution:$

$$Let\ f(x,y) = \frac{1}{(x+y)^2}$$

$(i) Range\ for\ x:\ 3\ to\ 4\ and\ h = 0.5$

$(ii) Range\ for\ y:\ 1\ to\ 2\ and\ k = 0.5$

y \ x	3	3.5	4
1	0.0625	0.0494	0.04
1.5	0.0494	0.04	0.0331
2	0.04	0.0331	0.0278

$$f(x,y) = \frac{1}{(x+y)^2}$$

$$f(3,1) = \frac{1}{(3+1)^2} = \frac{1}{16} = 0.0625$$

$$f(3.5,1) = \frac{1}{(3.5+1)^2} = \frac{1}{(4.5)^2} = 0.0494$$

$$f(4,1) = \frac{1}{(4+1)^2} = \frac{1}{25} = 0.04$$

$$I = \frac{hk}{4}[(Sum\ of\ \ four\ corners) + 2(Sum\ of\ nodes\ on\ boundary) + 4(Sum\ of\ interior\ nodes)]$$

$$I = \frac{(0.5)(0.5)}{4}[(0.0625 + 0.04 + 0.04 + 0.0278) + 2(0.0494 + 0.0494 + 0.0331 + 0.0331) + 4(0.04)]$$

$$I = \frac{0.25}{4}[(0.1703) + 0.330 + 0.16]$$

$$I = 0.0413$$

$$I = \frac{(0.5)(0.5)}{9}[(0.0625 + 0.04 + 0.04 + 0.0278) + 4(0.0494 + 0.0494 + 0.0331 + 0.0331) + 16(0.04)]$$

$$I = \frac{0.25}{9}[(0.1703) + 0.660 + 0.64]$$

$I = 0.0408$