LINEAR SYSTEMS

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogenate principles,

$$T [a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)]$$

$$\therefore T [a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

Example:

$y(t) = 2x(t)$

Solution:

$y_1 (t) = T[x_1(t)] = 2x_1(t)$

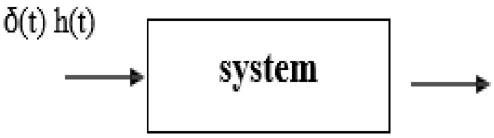
$y_2 (t) = T[x_2(t)] = 2x_2(t)$

$T [a_1 x_1(t) + a_2 x_2(t)] = 2[a_1 x_1(t) + a_2 x_2(t)]$

Which is equal to $a_1 y_1(t) + a_2 y_2(t)$. Hence the system is said to be linear.

Impulse Response:

The impulse response of a system is its response to the input δ(t) when the system is initially at rest. The impulse response is usually denoted h(t). In other words, if the input to an initially at rest system is δ(t) then the output is named h(t).



Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

If a system is both liner and time variant, then it is called liner time variant (LTV) system.

If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.

Response of a continuous-time LTI system and the convolution integral

(i) Impulse Response:

The ***impulse response*** h(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is δ(t), that is,

$$h(t)=T\{\delta(t)\} \text{ ------------(1)}$$

(ii) Response to an ArbitraryInput:

The input *x(*t) can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \qquad \text{-------- (2)}$$

Since the system is linear, the response *y(*t of the system to an arbitrary input x(t) can be expressed as

$$y(t) = \mathbf{T}\{x(t)\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau \qquad \text{- ------(3)}$$

Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\} \qquad \text{------- (4)}$$

Substituting **Eq.** (4) into **Eq.** (3), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \qquad \text{------ (5)}$$

Equation *(5)* indicates that a continuous-time LTI system is completely characterized by its impulse response *h(t)*.

(iii) ConvolutionIntegral:

Equation (5) defines the convolution of two continuous-time signals x (t) and h(t) denoted by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \qquad \text{------ (6)}$$

Equation (6) is commonly called the convolution integral. Thus, we have the fundamental result that the output of any continuous-time LTI system is the convolution of the input x (t) with the impulse response h(t) of the system. The following figure illustrates the definition of the impulse response h(t) and the relationship of Eq.(6).

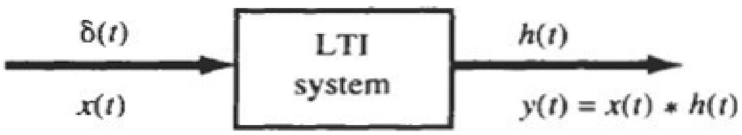


Fig.: Continuous-time LTI system.

(iv) Properties of the Convolution Integral:

The convolution integral has the following properties.

1. Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

2. Associative:

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

3. Distributive:

$$x(t) * \{h_1(t)\} + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

(v) **StepResponse:**

The step response s(t) of a continuous-time LTI system (represented by **T)** is defined to be the response of the system when the input is u(t); that is,

$$S(t) = T\{u(t)\}$$

In many applications, the step response s(t) is also a useful characterization of the system.

The step response s(t) can be easily determined by,

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

Thus, the step response s(t) can be obtained by integrating the impulse response h(t). Differentiating the above equation with respect to t, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Thus, the impulse response h(t) can be determined by differentiating the step response s(t).

Distortion less transmission through a system:

Transmission is said to be distortion-less if the input and output have identical wave shapes. i.e., in distortion-less transmission, the input x(t) and output y(t) satisfy the condition:

$$y(t) = Kx(t - t_d)$$

Where t_d = delay time and

k = constant.

Take Fourier transform on both sides

$$FT[y(t)] = FT[Kx(t - t_d)]$$

$$= K\,FT[x(t - t_d)]$$

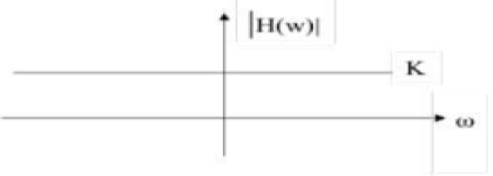
According to time shifting property,

$$Y(w) = KX(w)e^{-j\omega t_d}$$

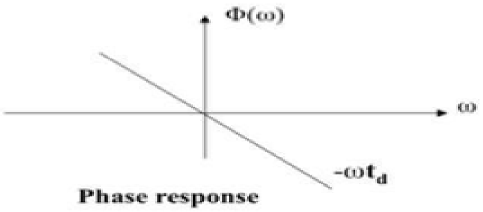
Thus, distortion less transmission of a signal x(t) through a system with impulse response h(t) is achieved when

$|H(\omega)|=K$ and (amplitude response)

$$\Phi(\omega) = -\omega t_d = -2\pi f t_d \quad phaseresponse$$



Amplitude response



Phase response