4.3 CONVOLUTION OF FOURIER TRANSFORM

Evaluate (a) $\int_0^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$ **(b)** $\int_0^\infty \frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$ **using Fourier transforms.**

Solution:

(a) Assume $f(x)=e^{-ax} \quad ; g(x)=e^{-bx}$

The Fourier sine transform f(x) is

$$F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s)=F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2} \text{ here } a=a; b=s$$

Similarly

$$\boxed{G_s(s)=F_s\left[e^{-bx}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_s(s)G_s(s)ds=\int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]ds=\int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2}{\pi}\int_0^\infty\left[\frac{s^2}{\left(a^2+s^2\right)\left(b^2+s^2\right)}\right]ds=\int_0^\infty e^{-ax-bx}dx$$

$$\int_0^\infty\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds=\frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$=\frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$=\frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$=\frac{-\pi}{2(a+b)}\left[e^{-\infty}-e^{-0}\right]_0^\infty$$

$$=\frac{-\pi}{2(a+b)}\left[0-1\right] \qquad \because e^{-\infty}=0; e^{-0}=1$$

$$\int_0^\infty\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds=\frac{\pi}{2(a+b)}$$

Put s=x we get

$$\int_0^\infty \left[\frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}\right]dx=\frac{\pi}{2(a+b)}$$

(b) Assume $f(x)=e^{-ax}\quad ;g(x)=e^{-bx}$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx\ .$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$F_c(s)=F_c\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\qquad \because \int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}\ \text{ here } a=a;b=s$$

Similarly

$$G_c(s)=F_c\left[e^{-bx}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]$$

We know that

$$\int_0^\infty F_c(s)G_c(s)ds=\int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]ds=\int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2ab}{\pi}\int_0^\infty\left[\frac{1}{\left(a^2+s^2\right)\left(b^2+s^2\right)}\right]ds=\int_0^\infty e^{-ax-bx}dx$$

$$\int_0^\infty\left[\frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds=\frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$$

$$=\frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$$

$$=\frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$=\frac{-\pi}{2ab(a+b)}\left[e^{-\infty}-e^{-0}\right]_0^\infty$$

$$=\frac{-\pi}{2ab(a+b)}\left[0-1\right]\qquad \because e^{-\infty}=0;e^{-0}=1$$

$$\int_0^\infty\left[\frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds=\frac{\pi}{2ab(a+b)}$$

Put s=x we get

$$\int_0^\infty\left[\frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}\right]dx=\frac{\pi}{2ab(a+b)}$$

Evaluate (a) $\int_0^\infty \frac{x^2}{(x^2+9)(x^2+16)}dx$ **, (b)** $\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx$ **using Fourier transforms.**

Solution:

(a) Assume $f(x) = e^{-ax} \quad ; g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2} \text{ here } a = a; b = s$$

Similarly

$$\boxed{G_s(s) = G_s\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_s(s)G_s(s)ds = \int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2}{\pi}\int_0^\infty\left[\frac{s^2}{(a^2+s^2)(b^2+s^2)}\right]ds = \int_0^\infty e^{-ax-bx}dx$$

$$\int_0^\infty\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]ds = \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}\left[e^{-\infty} - e^{-0}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}\left[0-1\right] \qquad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\boxed{\int_0^\infty\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]ds = \frac{\pi}{2(a+b)}} ------(1)$$

Put a=3 & b=4 and s=x we get

$$(1) \Rightarrow \int_0^\infty \frac{x^2}{(x^2+9)(x^2+16)}dx = \frac{\pi}{2(3+4)}$$

$$\boxed{\int_0^\infty \frac{x^2}{\left(x^2+9\right)\left(x^2+16\right)}dx = \frac{\pi}{14}}$$

(b) Assume $f(x) = e^{-ax} \quad ; g(x) = e^{-bx}$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\boxed{F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \quad \text{here } a=a; b=s$$

Similarly

$$\boxed{G_c(s) = F_c\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_c(s)G_c(s)ds = \int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2ab}{\pi}\int_0^\infty \left[\frac{1}{\left(a^2+s^2\right)\left(b^2+s^2\right)}\right]ds = \int_0^\infty e^{-ax-bx}dx$$

$$\int_0^\infty \left[\frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds = \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-\pi}{2ab(a+b)}\left[e^{-\infty}-e^{-0}\right]_0^\infty$$

$$= \frac{-\pi}{2ab(a+b)}\left[0-1\right] \qquad \because e^{-\infty}=0; e^{-0}=1$$

$$\boxed{\int_0^\infty \left[\frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds = \frac{\pi}{2ab(a+b)}}$$

Put a=1 & b=2 s=x we get

$$(1) \Rightarrow \int_0^\infty \frac{1}{\left(x^2+1\right)\left(x^2+4\right)}dx = \frac{\pi}{2(1)(2)(1+2)}$$

$$\boxed{\int_0^\infty \frac{1}{\left(x^2+1\right)\left(x^2+4\right)}dx = \frac{\pi}{12}}$$