4.4 Cosets

Define Left Coset and Right Coset of H in G.

Let $(H, *)$ be a subgroup of $(G, *)$.

For any $a \epsilon G$, the left coset of H, denoted by $a * H$, is the set

$a * H = \{a * h: h \epsilon H\}$ for all $a \epsilon G$

For any $a \epsilon G$, the right coset of H, denoted by $H * a$, is the set

$H * a = \{h * a: h \epsilon H\}$ for all $a \epsilon G$

Theorem: 1

Let $(H, *)$ be a subgroup of $(G, *)$. Then any two left Cosets (right Cosets) of H of a group $(G, *)$ are either identical or disjoint and the union of distinct left Cosets of H is G (or) The set of all distinct left Cosets of the subgroup H of the group $(G, *)$ forms a partition of G.

Proof:

Let $a, b \epsilon G$

Consider the Cosets $a * H$ and $b * H$

We shall prove that $a * H = b * H$ (or) $a * H \cap b * H = \emptyset$

Suppose $a * H \cap b * H \neq \emptyset$

Let c ϵ $a * H \cap b * H = \emptyset$

$\Rightarrow c \, \epsilon \, a * H$ and $c \, \epsilon \, b * H$

Let $c = a * h_1$ and $c = b * h_2$ for all $h_1, h_2 \, \epsilon \, H$

$\therefore a * h_1 = b * h_2$

Take h_1^{-1} on both sides

$\Rightarrow (a * h_1) * h_1^{-1} = (b * h_2) * h_1^{-1}$

$\Rightarrow a * (h_1 * h_1^{-1}) = b * (h_2 * h_1^{-1})$

$\Rightarrow a * e = b * h_3$ where $h_3 = h_2 * h_1^{-1}$

$\Rightarrow a = b * h_3$

$\Rightarrow a \in b * h_3$

$\Rightarrow a * H \subseteq b * H \, . \, . \, .$ (1)

IIIrly $b * H \subseteq a * H \, . \, . \, .$(2)

From (1) and (2) we have $a * H = b * H$

$\therefore$ Any two left cosets are either identical or distinct.

Each element of the left Coset $a * H$ is also an element of G.

∴ Every left coset of $a * H$ is a subset of G.

Hence $\bigcup_{a \epsilon G} a * H \subseteq G$ . . . (3)

If $a \epsilon G$, $a \epsilon a * H$ then $a \epsilon \bigcup_{a \epsilon G} a * H$

$G \subseteq \bigcup_{a \epsilon G} a * H$ . . . (4)

∴The set of all distinct left cosets of H is a partition "n' of the group G.

Hence the proof.

LAGRANGE'S THEOREM:

The order of a subgroup of a finite group is a divisor of the order of the group.

i.e., if H is a subgroup of a finite group $(G, *)$ then O(H) divides O(G).

Proof:

Let $(G, *)$ be a finite group of order n and H be a subgroup of G with order m.

$\Rightarrow O(H) = m$ & $O(G) = n$

We will prove that $O(H) / O(G)$

Since H contains m distinct elements, every left cost of H contains exactly m elements.

(Write the theorem: 1)

Let $a_1 * H, a_2 * H, \ldots, a_k * H$ be the distinct left cosets of H. Let $G = a_1 * H \cup a_2 * H \cup \ldots \cup a_k * H$

$O(G) = O(a_1 * H) + O(a_2 * H) + \ldots + O(a_k * H)$

$= O(H) + O(H) + \ldots + O(H)$

$= m + m + \ldots + m$ (n times)

$\Rightarrow n = mk$

$\Longrightarrow n/m = k$

$\Longrightarrow$ m divides n.

This means that $O(H) / O(G)$.

Hence the proof.

Normal Subgroup

A subgroup $(H,*)$ of $(G,*)$ is said to be normal subgroup of G, for $x \in G$ and for $h \in H$, if $x * h = h * x$ (or) for all $x \in G, xH = Hx$

Note:

Consider H as a subgroup of G, then the subgroup H is said to be normal, for all $x \in G, x * h * x^{-1} = H$(or) for all $x \in G, x * h * x^{-1} \in H$

Theorem: 1

Every subgroup of an abelian group is normal.

Proof:

Let $(G,*)$ be an abelian group and $(H,*)$ be a subgroup of G.

Let $x \in G$ be any element.

Then $xH = \{x * h / h \in H\}$

$= \{h * x / h \in H\}$ (G is abelian)

$= Hx$

Since "x" is arbitrary, $xH = Hx \ \forall \ x \in G$

Hence H is a normal subgroup of G.

Hence the proof.

Theorem: 2

Prove that intersection of two normal subgroup of $(G,*)$ is a normal subgroup of $(G,*)$.

Proof:

Let $(H,*)$ and $(K,*)$are two normal subgroup.

⇒ H and K are subgroups of G.

⇒ $H \cap K$ is a subgroup of G. (Already proved)

To prove $(H \cap K,\ *)$ is a normal subgroup of $(G,*)$.

Let $h \in\ H \cap K$ be any element and $x \in G$ be any element.

Then $x \in G$ and$h \in H$ and $h \in K$

Since H and K are normal, $x\ * h\ * x^{-1} \in\ H$ . . . (1)

and $x *\ h *\ x^{-1} \in\ K$ . . . (2)

From (1) and (2) we get,

$$x\ * h\ * x^{-1} \in\ H\ \cap K$$

Hence $H\ \cap K$ is a normal subgroup of G.

Hence the proof.