

## UNIT – I – RANDOM VARIABLES

### Random Experiment

An experiment whose output is uncertain even though all the outcomes are known.

**Example:** Tossing a coin, Throwing a fair die, Birth of a baby.

### Sample Space:

The set of all possible outcomes in a random experiment. It is denoted by  $S$ .

### Example:

For tossing a fair coin,  $S = \{H, T\}$

For throwing a fair die,  $S = \{1, 2, 3, 4, 5, 6\}$

For birth of a baby,  $S = \{M, F\}$

### Event:

A subset of sample space is event. It is denoted by  $A$ .

### Mutually Exclusive Events:

Two events  $A$  and  $B$  are said to be mutually exclusive events if they do not occur simultaneously. If  $A$  and  $B$  are mutually exclusive, then  $A \cap B = \Phi$

### Example:

Tossing two unbiased coins  $S = \{HH, HT, TH, TT\}$

(i) Let  $A = \{HH\}, B = \{HT\}$

$$A \cap B = \{H\} \neq \Phi$$

Then A and B are not mutually exclusive.

(i) Let  $A = \{HH\}, B = \{TT\}$

$$A \cap B = \Phi$$

Then A and B are mutually exclusive.

**Probability:**

Probability of an event A is  $P(A) = \frac{n(A)}{n(S)}$

i.e.,  $P(A) = \frac{\text{number of cases favourable to A}}{\text{Total number of cases}}$

**Axioms of Probability:**

(i)  $0 \leq P(A) \leq 1$

(ii)  $P(S) = 1$

(iii)  $P(A \cup B) = P(A) + P(B)$ , if A and B are mutually exclusive.

**Note:**

(i)  $P(\phi) = 0$

(ii)  $P(\overline{A}) = 1 - P(A)$ , for any event A

(iii)  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ , for any two events A and B.

### Independent events:

Two events A and B are said to be independent if occurrence of A does not affect the occurrence of B.

Condition for two events A and B are independent:

$$P(A \cap B) = P(A) P(B)$$

### Conditional Probability:

If the probability of the event A provided the event B has already occurred is called the conditional probability and is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

The probability of an event B provided A has occurred already is given by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

### Random Variables:

A random variable is a function that assigns a real number for all the outcomes in the sample space of a random experiment.

### Example:

Toss two coins then the sample space  $S = \{HH, HT, TH, TT\}$

Now we define a random variable X to denote the number of heads in 2 tosses.

$$X(HH) = 2$$

$$X(HT) = 1$$

$$X(TH) = 1$$

$$X(TT) = 0$$

**Types of Random Variables:**

- (i) Discrete Random Variables
- (ii) Continuous Random Variables

**Probability mass function (PMF):**

Let  $X$  be discrete random variable. Then  $P(X = x_i) = p(x_i) = p_i$  is said to be a Probability mass function of  $X$ , if

- (i)  $0 \leq p(x_i) \leq 1$
- (ii)  $\sum_i p(x_i) = 1$

The collection of pairs  $\{x_i, p_i\}, i = 1, 2, 3, \dots$  is called the probability distribution of the random variable  $X$ , which is sometimes in the form of a table as given below:

$X = x_i$	$x_1$	$x_2$	$\dots$	$x_r$	$\dots$
$P(X = x_i)$	$p_1$	$p_2$	$\dots$	$p_r$	$\dots$

**Problems on Discrete Random Variables**

**1.A Discrete Random Variable  $X$  has the following probability distribution**

<b>X</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>
<b>P(x)</b>	<b>a</b>	<b>3a</b>	<b>5a</b>	<b>7a</b>	<b>9a</b>	<b>11a</b>	<b>13a</b>	<b>15a</b>	<b>17a</b>

- (i) Find the value of “a”.
- (ii) Find  $P[X < 3]$ ,  $P[0 < X < 3]$ ,  $P[X \geq 3]$
- (iii) Find the distribution of X.

**Solution:**

(i) We know that  $\sum P(x) = 1$

$$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$\Rightarrow 81a = 1$$

$$\Rightarrow a = \frac{1}{81}$$

(ii)  $P[X < 3] = P[X = 0] + P[X = 1] + P[X = 2]$

$$= a + 3a + 5a$$

$$= 9a$$

$$= \frac{9}{81}$$

$$P[0 < X < 3] = P[X = 1] + P[X = 2]$$

$$= 3a + 5a$$

$$= 8a$$

$$= \frac{8}{81}$$

$$P[X \geq 3] = 1 - P[X < 3]$$

$$= 1 - \frac{9}{81}$$

$$= \frac{72}{81}$$

**(iii) Distribution of X:**

<b>X</b>	<b>P(x)</b>	<b>F(X) = P[X ≤ x]</b>
0	a	$F(0) = P[X \leq 0] = \frac{1}{81}$
1	3a	$F(1) = P[X \leq 1] = F(0) + P(1) = \frac{1}{81} + \frac{3}{81} = \frac{4}{81}$
2	5a	$F(2) = P[X \leq 2] = F(1) + P(2) = \frac{4}{81} + \frac{5}{81} = \frac{9}{81}$
3	7a	$F(3) = P[X \leq 3] = F(2) + P(3) = \frac{9}{81} + \frac{7}{81} = \frac{16}{81}$
4	9a	$F(4) = P[X \leq 4] = F(3) + P(4) = \frac{16}{81} + \frac{9}{81} = \frac{25}{81}$
5	11a	$F(5) = P[X \leq 5] = F(4) + P(5) = \frac{25}{81} + \frac{11}{81} = \frac{36}{81}$
6	13a	$F(6) = P[X \leq 6] = F(5) + P(6) = \frac{36}{81} + \frac{13}{81} = \frac{49}{81}$
7	15a	$F(7) = P[X \leq 7] = F(6) + P(7) = \frac{49}{81} + \frac{15}{81} = \frac{64}{81}$
8	17a	$F(8) = P[X \leq 8] = F(7) + P(8) = \frac{64}{81} + \frac{17}{81} = \frac{81}{81}$

**2. A Discrete Random Variable X has the following probability distribution**

<b>X</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>
<b>P(x)</b>	<b>0</b>	<b>k</b>	<b>2k</b>	<b>2k</b>	<b>3k</b>	<b>k<sup>2</sup></b>	<b>2k<sup>2</sup></b>	<b>7k<sup>2</sup> + k</b>

- (i) Find the value of “k”.
- (ii) Find  $P[X < 6]$ ,  $P[1 < X < 5]$ ,  $P[X \geq 6]$ ,  $P[X > 2]$
- (iii) Find  $P[1.5 < X < 4.5 / X > 2]$
- (iv) Find the distribution of X and find the value of k if  $P[X < k] > \frac{1}{2}$

**Solution:**

(i) We know that  $\sum P(x) = 1$

$$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow 10k^2 + 9k = 1$$

$$\Rightarrow 10k^2 + 9k - 1 = 0$$

$$\Rightarrow (k + 1)(10k - 1) = 0$$

$$\Rightarrow k = -1 \text{ (or) } k = \frac{1}{10}$$

(ii)  $P[X \geq 6] = P[X = 6] + P[X = 7]$

$$= 2k^2 + 7k^2 + k$$

$$= 9k^2 + k$$

$$= \frac{9}{100} + \frac{1}{10}$$

$$= \frac{19}{100}$$

(iii)  $P[X < 6] = 1 - P[X \geq 6]$

$$= 1 - \frac{19}{100}$$

$$= \frac{81}{100}$$

(iv)  $P[1 < X < 5] = P[X = 2] + P[X = 3] + P[X = 4]$

$$= 2k + 2k + 3k$$

$$= 7k$$

$$= \frac{7}{10}$$

(v)  $P[1.5 < X < 4.5 / X > 2] = \frac{P[1.5 < X < 4.5 \cap X > 2]}{P[X > 2]}$

$$= \frac{P[2 < X < 4.5]}{P[X > 2]}$$

$$= \frac{P[X=3] + P[X=4]}{P[X > 2]}$$

$$= \frac{\frac{5}{10}}{\frac{7}{10}}$$

$$= \frac{5}{7}$$

**Distribution of X:**

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X	P(x)	F(X) = P[X ≤ x]
0	0	F(0) = P[X ≤ 0] = 0
1	k	F(1) = P[X ≤ 1] = F(0) + P(1) = 0 + $\frac{1}{10}$ = $\frac{1}{10}$



2	$2k$	$F(2) = P[X \leq 2] = F(1) + P(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$
3	$2k$	$F(3) = P[X \leq 3] = F(2) + P(3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10}$
4	$3k$	$F(4) = P[X \leq 4] = F(3) + P(4) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10}$
5	$k^2$	$F(5) = P[X \leq 5] = F(4) + P(5) = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$
6	$2k^2$	$F(6) = P[X \leq 6] = F(5) + P(6) = \frac{81}{100} + \frac{2}{100} = \frac{83}{100}$
7	$7k^2 + k$	$F(7) = P[X \leq 7] = F(6) + P(7) = \frac{83}{100} + \frac{7}{100} + \frac{1}{10} = \frac{100}{100}$

The value of  $k = 4$  when  $P[X < k] > \frac{1}{2}$

**3. If the random variable  $X$  takes the values 1, 2, 3 and 4 such that  $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$ . Find the probability distribution.**

**Solution:**

Let  $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4) = k$

$$\Rightarrow 2P(X = 1) = k$$

$$\Rightarrow P(X = 1) = \frac{k}{2}$$

$$\Rightarrow 3P(X = 2) = k$$

$$\Rightarrow P(X = 2) = \frac{k}{3}$$

$$\Rightarrow P(X = 3) = k$$

$$\Rightarrow 5P(X = 3) = k$$

$$\Rightarrow P(X = 3) = \frac{k}{5}$$

We know that  $\sum P(x) = 1$

$$\Rightarrow P(1) + P(2) + P(3) + P(4) = 1$$

$$\Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\Rightarrow \frac{15k + 10k + 30k + 6k}{30} = 1$$

$$\Rightarrow \frac{61k}{30} = 1$$

$$\Rightarrow k = \frac{30}{61}$$

The Probability Distribution is

X	1	2	3	4
P(x)	$\frac{k}{2} = \frac{1}{2} \times \frac{30}{61} = \frac{15}{61}$	$\frac{k}{3} = \frac{1}{3} \times \frac{30}{61} = \frac{10}{61}$	$k = \frac{30}{61}$	$\frac{k}{5} = \frac{1}{5} \times \frac{30}{61} = \frac{6}{61}$

**4. Suppose that the random variable X assumes three values 0, 1 and 2 with probabilities 1/3, 1/6 and 1/2 respectively. Obtain the distribution function of X.**

**Solution:**

Values of $X = x$	0	1	2
$P(x)$	$1/3$	$1/6$	$1/2$
	$P(0)$	$P(1)$	$P(2)$

**The distribution of X**

<b>X</b>	<b>P(x)</b>	<b>F(X) = P[X ≤ x]</b>
0	$1/3$	$F(0) = P[X ≤ 0] = \frac{1}{3}$
1	$1/6$	$F(1) = P[X ≤ 1] = F(0) + P(1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$
2	$1/2$	$F(2) = P[X ≤ 2] = F(1) + P(2) = \frac{1}{2} + \frac{1}{2} = 1$

**Mathematical expectation for discrete random variable ★**

**Note:**

(i)  $E(c) = c$

(ii)  $Var(c) = 0$

(iii)  $E(aX) = aE(X)$

(iv)  $E(aX + b) = aE(X) + b$

(v)  $Var(aX) = a^2Var(X)$

(vi)  $Var(aX ± b) = a^2Var(X)$

**Problems:**

If  $Var(X) = 4$ , find  $Var(4X + 5)$ , where  $X$  is a random variable.

**Solution:**

We know that  $Var(aX + b) = a^2Var(X)$

Here  $a = 4, Var(X) = 4$

$$Var(4X + 5) = 4^2Var(X) = 16 \times 4 = 64$$

**Continuous Random Variable:**

If  $X$  is a random variable which can take all the values in an interval then  $X$  is called continuous random variable.

**Properties of Probability Density Function:**

The Probability density function of the random variable  $X$  denoted by  $f(x)$  has the following properties.

(i)  $f(x) \geq 0$

(ii)  $\int_{-\infty}^{\infty} f(x)dx = 1$

**Cumulative Distribution Function (CDF):**

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f(x)dx$$

**Properties of CDF:**

(i)  $F(-\infty) = 0$

(ii)  $F(\infty) = 1$

(iii)  $\frac{d}{dx}[F(x)] = f(x)$

(iv)  $P(X \leq a) = F(a)$

(v)  $P(X > a) = 1 - F(a)$

(vi)  $P(a \leq X \leq b) = F(b) - F(a)$

**Problems on Continuous Random Variables:**

1. A continuous random variable X has a density function  $f(x) = \frac{K}{1+x^2}$ ,  $-\infty \leq$

$X \leq \infty$ . Find the values of K.

**Solution:**

We know that  $\int_{-\infty}^{\infty} f(x) dx = 1$

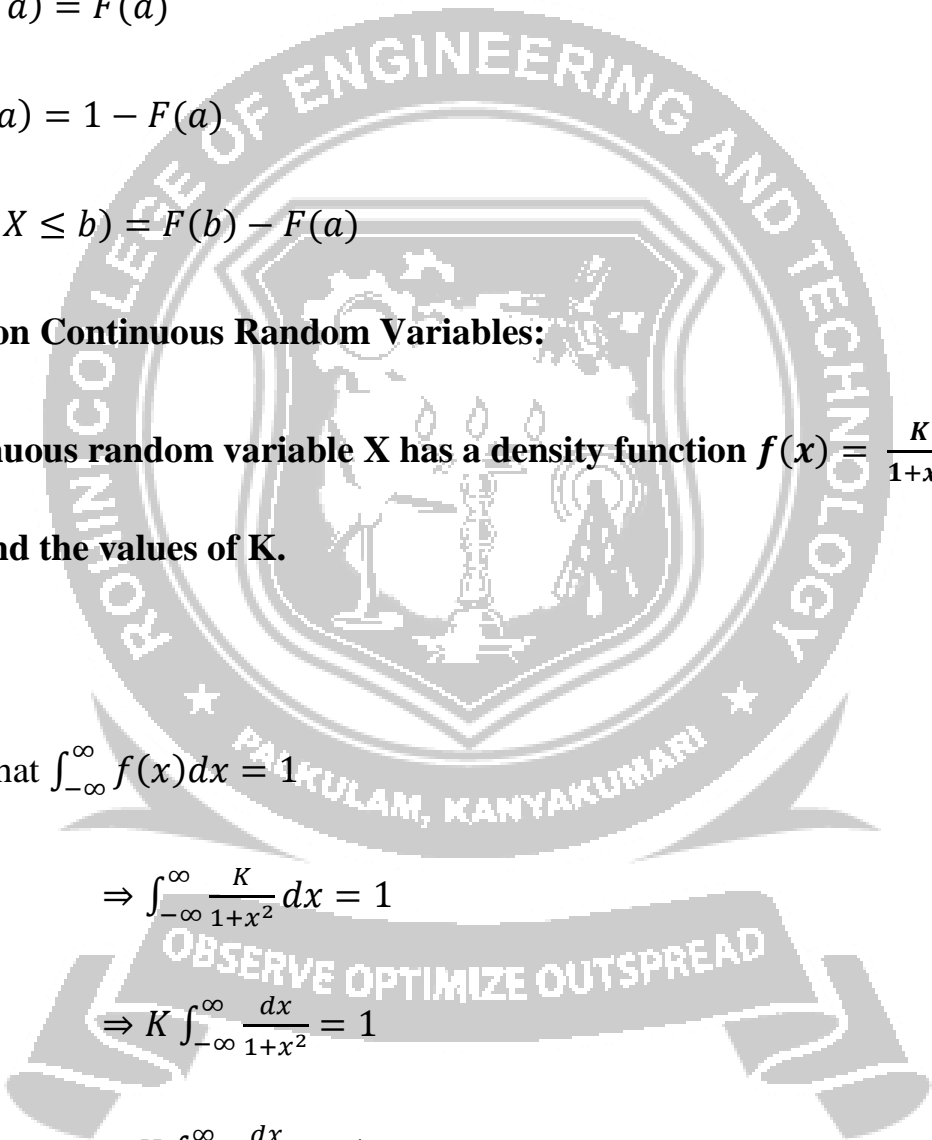
$$\Rightarrow \int_{-\infty}^{\infty} \frac{K}{1+x^2} dx = 1$$

$$\Rightarrow K \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$\Rightarrow K \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$\Rightarrow K[\tan^{-1}x]_{-\infty}^{\infty} = 1$$

$$\Rightarrow K[\tan^{-1}\infty - \tan^{-1}(-\infty)] = 1$$



$$\Rightarrow K \left[ \frac{\pi}{2} + \frac{\pi}{2} \right] = 1$$

$$\Rightarrow K \left[ \frac{2\pi}{2} \right] = 1$$

$$\Rightarrow K = \frac{1}{\pi}$$

2. If a random variable X has PDF  $f(x) = \begin{cases} \frac{1}{4}, & |x| < 2 \\ 0, & |x| > 2 \end{cases}$  Find (i)  $P[X < 1]$

(ii)  $P[|X| > 1]$  (iii)  $P[2X + 3 > 5]$

**Solution:**

$$(i) \quad P[X < 1] = \int_{-2}^1 f(x) dx$$

$$= \int_{-2}^1 \frac{1}{4} dx$$

$$= \frac{1}{4} [x]_{-2}^1$$

$$= \frac{1}{4} [1 - (-2)]$$

$$= \frac{3}{4}$$

$$(ii) \quad P[|X| > 1] = 1 - P[-1 < X < 1]$$

$$= 1 - \int_{-1}^1 f(x) dx$$

$$= 1 - \int_{-1}^1 \frac{1}{4} dx$$

$$= 1 - \frac{1}{4} [x]_{-1}^1$$

$$= 1 - \frac{1}{4} [1 - (-1)]$$

$$= 1 - \frac{2}{4}$$

$$= \frac{2}{4}$$

(iii)  $P[2X + 3 > 5] = P[2X > 5 - 3]$

$$= P\left[X > \frac{5-3}{2}\right]$$

$$= P\left[X > \frac{2}{2}\right]$$

$$= P[X > 1]$$

$$= \int_1^2 f(x) dx$$

$$= \int_1^2 \frac{1}{4} dx$$

$$= \frac{1}{4} [x]_1^2$$

$$= \frac{1}{4} [2 - (1)] = \frac{1}{4}$$

**Mathematical expectation of continuous random variables**

(i)  $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

(ii)  $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

(iii)  $Var(X) = E(X^2) - E[(X)]^2$

**Problems:**

**1. Let X be a continuous random variable with probability density function**

**$f(x) = kx(2 - x), 0 < x < 2$ . Find (i) k (ii) mean (iii) variance (iv) cumulative**

**distribution function of X (v) rth moment.**

**Solution:**

**(i) To find k**

$$\int_0^2 f(x) dx = 1 \Rightarrow k \int_0^2 (2x - x^2) dx = 1$$

$$\Rightarrow k \left[ \frac{2x^2}{2} - \frac{x^3}{3} \right]_0^2 = 1$$

$$\Rightarrow k \left[ 4 - \frac{8}{3} \right] = 1$$

$$\Rightarrow k \left( \frac{4}{3} \right) = 1$$

$$\Rightarrow k = \frac{3}{4}$$

**(ii) To calculate mean of X**

$$E(X) = \int_0^2 x f(x) dx$$

$$= \int_0^2 x^2 \frac{3}{4} (2 - x) dx$$

$$= \frac{3}{4} \int_0^2 (2x^2 - x^3) dx$$

$$= \frac{3}{4} \left[ \frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2$$

$$= \frac{3}{4} \left( \frac{16}{3} - 4 \right)$$

$$= \frac{3}{4} \times \frac{4}{3} = 1$$

**(iii) To calculate variance of X**



$$\begin{aligned}
 E(X^2) &= \int_0^2 x^2 f(x) dx \\
 &= \int_0^2 x^3 \frac{3}{4} (2-x) dx \\
 &= \frac{3}{4} \int_0^2 (2x^3 - x^4) dx \\
 &= \frac{3}{4} \left[ \frac{2x^4}{4} - \frac{x^5}{5} \right]_0^2 \\
 &= \frac{3}{4} \left( 8 - \frac{32}{5} \right) \\
 &= \frac{3}{4} \times \frac{8}{5} = \frac{6}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(X) &= E(X^2) - E[X]^2 \\
 &= \frac{6}{5} - 1 = \frac{1}{5}
 \end{aligned}$$

(iv) To calculate CDF of X

$$\begin{aligned}
 F_X(x) &= P(X \leq x) = \int_{-\infty}^x f(x) dx \\
 &= \int_0^x f(x) dx \\
 &= \int_0^x \frac{3}{4} x(2-x) dx \\
 &= \frac{3}{4} \int_0^x (2x - x^2) dx \\
 &= \frac{3}{4} \left[ \frac{2x^2}{2} - \frac{x^3}{3} \right]_0^x \\
 &= \frac{3}{4} \left( x^2 - \frac{x^3}{3} \right)
 \end{aligned}$$

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$$= \frac{1}{4}(3x^2 - x^3)$$

$$F(x) = \begin{cases} 0; & x < 0 \\ \frac{1}{4}(3x^2 - x^3); & 0 \leq x < 2 \\ 1; & x \geq 2 \end{cases}$$

(v) To find the rth moment:

$$\begin{aligned} E(x^r) &= \int_{-\infty}^{\infty} x^r f(x) dx \\ &= \int_0^2 x^r \frac{3}{4} x(2-x) dx \\ &= \frac{3}{4} \int_0^2 (2x^{r+1} - x^{r+2}) dx \\ &= \frac{3}{4} \left[ \frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3} \right]_0^2 \\ &= \frac{3}{4} \left[ \left( 2 \frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3} \right) - (0 - 0) \right] \\ &= \frac{3}{4} \times 2^r 2^2 \left[ \frac{1}{r+2} - \frac{1}{r+3} \right] \end{aligned}$$

$$= 6 \cdot \frac{2^r}{(r+2)(r+3)}$$

2. The probability distribution function of a random variable X is

$$f(x) = \begin{cases} x; & 0 < x < 1 \\ 2 - x; & 1 < x < 2 \\ 0; & x > 2 \end{cases}$$

Find the cdf of X.

**Solution:**

We know that c.d.f  $F(x) = \int_{-\infty}^x f(x) dx$

(i) When  $0 < x < 1$

$$F(x) = \int_{-\infty}^0 f(x)dx + \int_0^x f(x)dx$$

$$= 0 + \int_0^x x dx$$

$$= \left[ \frac{x^2}{2} \right]_0^x$$

$$= \frac{x^2}{2} - 0$$

$$= \frac{x^2}{2}$$

(ii) When  $1 < x < 2$

$$F(x) = \int_{-\infty}^0 f(x)dx + \int_0^1 f(x)dx + \int_1^x f(x)dx$$

$$= 0 + \int_0^1 f(x)dx + \int_1^x x dx$$

$$= \left[ \frac{x^2}{2} \right]_0^1 + \left[ 2x - \frac{x^2}{2} \right]_1^x$$

$$= \left( \frac{1}{2} - 0 \right) + \left[ \left( 2x - \frac{x^2}{2} \right) - \left( 2 - \frac{1}{2} \right) \right]$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

(iii) When  $x > 2$

$$F(x) = \int_{-\infty}^0 f(x)dx + \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^x f(x)dx$$

$$= 0 + \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^x x dx$$

$$= 0 + \int_0^1 x dx + \int_1^2 (2 - x) dx + 0$$

$$= \left[ \frac{x^2}{2} \right]_0^1 + \left[ 2x - \frac{x^2}{2} \right]_1^2$$

$$= \left( \frac{1}{2} - 0 \right) + \left[ (4 - 2) - \left( 2 - \frac{1}{2} \right) \right]$$

$$= \frac{1}{2} + 2 - \frac{3}{2} = 1$$

$$F(x) = \begin{cases} \frac{x^2}{2}; & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1; & 1 < x < 2 \\ 1; & x > 2 \end{cases}$$

3. The Cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0; & x < 0 \\ x^2; & 0 \leq x < \frac{1}{2} \\ 1 - \frac{3}{25} (3 - x)^2; & \frac{1}{2} \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Find the pdf of X and evaluate  $P(|X| \leq 1)$  using both pdf and cdf.

**Solution:**

Given

$$F(x) = \begin{cases} 0; & x < 0 \\ x^2; & 0 \leq x < \frac{1}{2} \\ 1 - \frac{3}{25} (3 - x)^2; & \frac{1}{2} \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

PDF is  $f(x) = \frac{d}{dx}[F(x)]$

$$f(x) = \begin{cases} 0; & x < 0 \\ 2x; & 0 \leq x < \frac{1}{2} \\ \frac{6}{25}(3-x); & \frac{1}{2} \leq x < 3 \\ 0, & x \geq 3 \end{cases}$$

To find  $P(|X| \leq 1)$  using cdf:

$$\begin{aligned} P(|X| \leq 1) &= P(-1 \leq X \leq 1) \\ &= F(1) - F(-1) \\ &= \left[1 - \frac{3}{25}(3-1)^2\right] - 0 \\ &= 1 - \frac{12}{25} \\ &= \frac{25-12}{25} = \frac{13}{25} \end{aligned}$$

To find  $P(|X| \leq 1)$  using pdf:

$$\begin{aligned} P(|X| \leq 1) &= P(-1 \leq X \leq 1) \\ &= \int_{-1}^1 f(x) dx \\ &= \int_{-1}^0 f(x) dx + \int_0^{\frac{1}{2}} f(x) dx + \int_{\frac{1}{2}}^1 f(x) dx \end{aligned}$$

$$\begin{aligned}
 &= 0 + \int_0^{\frac{1}{2}} 2x \, dx + \int_{\frac{1}{2}}^1 \frac{6}{25} (3 - x) \, dx \\
 &= 2 \left( \frac{x^2}{2} \right)_0^{\frac{1}{2}} + \frac{6}{25} \left[ 3x - \frac{x^2}{2} \right]_{\frac{1}{2}}^1 \\
 &= \frac{1}{4} + \frac{6}{25} \left[ 3 - \frac{1}{2} - \frac{3}{2} + \frac{1}{8} \right] \\
 &= \frac{13}{25}
 \end{aligned}$$

**Moment Generating Function:**

The moment generating function (MGF) of a random variable “X” (about origin) whose probability function f(x) is given by  $M_X(t) = E(e^{tx})$

$$= \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x) \, dx, & \text{for a continuous random variable} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x), & \text{for a discrete probability distribution} \end{cases}$$

**Problems:**

1. If a random variable “X” has the MGF,  $M_X(t) = \frac{2}{2-t}$ , find the variance of X.

**Solution:**

Given  $M_X(t) = \frac{2}{2-t} = 2(2-t)^{-1}$

$$M_X'(t) = -2(2-t)^{-2}(-1)$$

$$= 2(2 - t)^{-2}$$

$$M_X'(t = 0) = 2(2 - 0)^{-2} = \frac{2}{4} = \frac{1}{2}$$

$$M_X''(t) = -4(2 - t)^{-3}(-1)$$

$$= 4(2 - t)^{-3}$$

$$M_X''(t = 0) = 4(2 - 0)^{-3} = \frac{4}{8} = \frac{1}{2}$$

$$\text{Var}(X) = E(X^2) - E[X]^2$$

$$= \frac{1}{2} - \left(\frac{1}{4}\right)^2 = \frac{1}{4}$$

### Moments

The rth moment about origin is  $\mu_r' = E[x_r']$

First moment about origin  $\mu_1' = E[X] = E(X^2) - [E(X)]^2$

Variance  $\sigma^2 = \mu_2' - (\mu_1')^2$

The rth moment about mean is  $\mu_r = E[(X - \mu)^r]$ , where  $\mu$  is mean of  $X$ .

$$\Rightarrow \mu_1 = E[(X - \mu)^1]$$

$$= E[X] - E[\mu] = \mu - \mu = 0$$

$$\Rightarrow \mu_1 = 0$$

$$\Rightarrow \mu_2 = E[(X - \mu)^2]$$

$$\begin{aligned}
 &= E[X^2 + \mu^2 - 2X\mu] \\
 &= E[X^2] + \mu^2 - 2E[X]\mu \\
 &= E(X^2) + [E(X)]^2 - 2E(X)E(X) \\
 &= E(X^2) + [E(X)]^2 - 2[E(X)]^2 \\
 &= E(X^2) - [E(X)]^2 = \sigma^2
 \end{aligned}$$

$$\Rightarrow \mu_2 = \sigma^2$$

1. If the probability density of X is given  $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$  Find its  $r^{\text{th}}$  moment about origin. Hence find evaluate  $E[(2X + 1)^2]$

**Solution:**

The  $r^{\text{th}}$  moment about origin is given by

$$\begin{aligned}
 \mu'_r &= E[x_r'] = \int_0^1 x^r f(x) dx \\
 &= \int_0^1 x^r 2(1-x) dx \\
 &= 2 \int_0^1 (x^r - x^{r+1}) dx \\
 &= 2 \left[ \frac{x^{r+1}}{r+1} - \frac{x^{r+1+1}}{r+2} \right]_0^1 \\
 &= 2 \left[ \frac{1}{r+1} - \frac{1}{r+2} \right]
 \end{aligned}$$



$$= 2 \left[ \frac{(r+2)-(r+1)}{(r+2)(r+1)} \right] = \frac{2}{r^2+3r+2}$$

$$\begin{aligned} E[(2X + 1)^2] &= E[4X^2 + 4X + 1] \\ &= 4E[X^2] + 4E[X] + 1 \\ &= 4\mu'_2 + 4\mu'_1 + 1 \\ &= 4 \frac{2}{2^2+3(2)+2} + 4 \frac{2}{2^2+3(2)+2} + 1 \\ &= \frac{8}{12} + \frac{8}{6} + 1 = 3 \end{aligned}$$

$$\therefore E[(2X + 1)^2] = 3$$

**Moment Generating Function (MGF)**

Let  $X$  be a random variable. Then the MGF of  $X$  is  $M_X(t) = E[e^{tx}]$

If  $X$  is a discrete random variable, then the MGF is given by

$$M_X(t) = \sum_x e^{tx} p(x)$$

If  $X$  is a continuous random variable, then the MGF is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

**Define MGF and why it is called so?**

**Solution:**

Let  $X$  be a random variable. Then the MGF of  $X$  is  $M_X(t) = E[e^{tX}]$ .

Let  $X$  be a continuous random variable. Then

$$\begin{aligned}
 M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_{-\infty}^{\infty} \left[ 1 + \frac{tx}{1!} + \frac{t^2 x^2}{2!} + \dots + \frac{t^r x^r}{r!} + \dots \right] f(x) dx \\
 &= \int_{-\infty}^{\infty} \left[ f(x) + \frac{tx}{1!} f(x) + \frac{t^2 x^2}{2!} f(x) + \dots + \frac{t^r x^r}{r!} f(x) + \dots \right] dx \\
 &= \int_{-\infty}^{\infty} f(x) dx + \frac{t}{1!} \int_{-\infty}^{\infty} x f(x) dx + \frac{t^2}{2!} \int_{-\infty}^{\infty} x^2 f(x) dx \dots + \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r f(x) dx + \dots
 \end{aligned}$$

$$M_X(t) = 1 + \frac{t}{1!} \mu'_1 + \frac{t^2}{2!} \mu'_2 + \dots + \frac{t^r}{r!} \mu'_r + \dots$$

$\therefore M_X(t)$  generates moments therefore it is moment generation function

**Note:**

If  $X$  is a discrete RV and if  $M_X(t)$  is known, then  $\mu'_r = \left[ \frac{d^r}{dt^r} [M_X(t)] \right]_{t=0}$

If  $X$  is a continuous RV and if  $M_X(t)$  is known, then  $\mu'_r$   
 $= r! \times \text{coeff of } t^r \text{ in } M_X(t)$

**Problems under MGF of discrete random variable**

$$M_X(t) = \sum_x e^{tx} p(x)$$

If  $X$  is a discrete RV and if  $M_X(t)$  is known, then  $\mu'_r = \left[ \frac{d^r}{dt^r} [M_X(t)] \right]_{t=0}$

**1. Let  $X$  be the number occur when a die is thrown. Find the MGF mean and variance of  $X$ .**

**Solution:**

$x$	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$\begin{aligned}
 \text{(i) } M_X(t) &= \sum_{x=1}^6 e^{tx} p(x) \\
 &= e^t P(1) + e^{2t} P(2) + e^{3t} P(3) + e^{4t} P(4) + e^{5t} P(5) + e^{6t} P(6) \\
 &= e^t \frac{1}{6} + e^{2t} \frac{1}{6} + e^{3t} \frac{1}{6} + e^{4t} \frac{1}{6} + e^{5t} \frac{1}{6} + e^{6t} \frac{1}{6}
 \end{aligned}$$

$$M_X(t) = \frac{1}{6} [e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$$

$$\text{(ii) } E(X) = \left[ \frac{d}{dt} M_X(t) \right]_{t=0} = \frac{1}{6} [e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$$

$$= \frac{1}{6} [1 + 2 + 3 + 4 + 5 + 6] = \frac{21}{6}$$

$$\Rightarrow E(X) = 3.5$$

$$E(X^2) = \left[ \frac{d^2}{dt^2} [M_X(t)] \right]_{t=0}$$

$$= \frac{1}{6} [e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]$$

$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

$$= 15.1$$

(iii) Variance of  $X = E(X^2) - [E(X)]^2 = 15.1 - 12.25$

$$\sigma_x = 2.85$$

**2. Find the moment generating function for the distribution**

where  $(X = x) = \begin{cases} \frac{2}{3}; & x = 1 \\ \frac{1}{3}; & x = 2 \\ 0; & \text{otherwise} \end{cases}$ . Also find its mean & variance,

**Solution:**

The probability distribution of  $X$  is given by

$x$	1	2
$p(x)$	2/3	1/3

$$\Rightarrow M_X(t) = E[e^{tx}] = \sum_{x=1}^2 e^{tx} p(x)$$

$$= e^t p(X = 1) + e^{2t} p(X = 2) = e^t \frac{2}{3} + e^{2t} \frac{1}{3}$$

$$\Rightarrow M_X(t) = \frac{1}{3}(2e^t + e^{2t})$$

$$\Rightarrow E(X) = M'_X(0)$$

$$= \left[ \frac{d}{dt} \left[ \frac{1}{3}(2e^t + e^{2t}) \right] \right]_{t=0}$$

$$= \frac{1}{3}(2e^t + 2e^{2t})$$

$$\Rightarrow E(X) = \frac{4}{3}$$

$$\Rightarrow E(X^2) = M_X''(0) = \left[ \frac{d^2}{dt^2} \left[ \frac{1}{3}(2e^t + e^{2t}) \right] \right]_{t=0}$$

$$= \left[ \frac{d}{dt} \left[ \frac{1}{3}(2e^t + 2e^{2t}) \right] \right]_{t=0}$$

$$= \left[ \frac{1}{3}(2e^t + 4e^{2t}) \right]_{t=0} = \frac{6}{3} = 2$$

$$\text{Variance of } X = E(X^2) - [E(X)]^2 = 2 - \left(\frac{4}{3}\right)^2$$

$$\Rightarrow \text{Var}(X) = \frac{2}{9}$$

**3. Let  $X$  be a RV with PMF  $P(x) = \left(\frac{1}{2}\right)^x$ ;  $x = 1, 2, 3, \dots$  Find MGF and hence find mean and variance of  $X$ .**

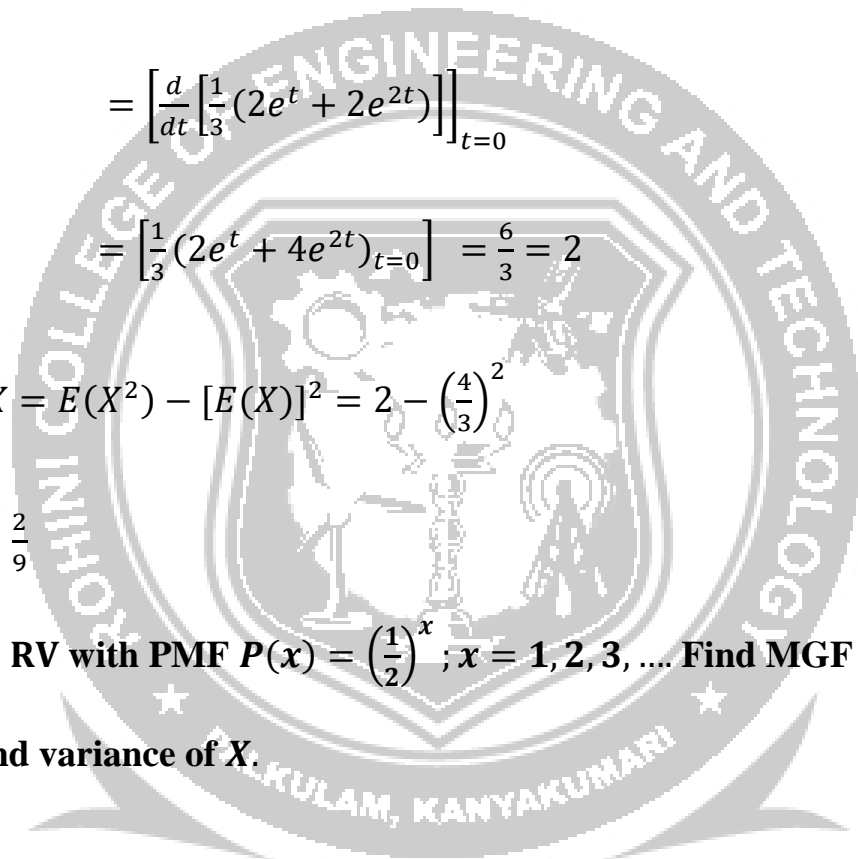
**Solution:**

$$(i) M_X(t) = E[e^{tX}]$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \left(\frac{1}{2}\right)^x$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$



OBSERVE OPTIMIZE OUTSPREAD

$$= \left[ \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots \right]$$

$$= \frac{e^t}{2} \left( 1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \dots \right)$$

$$= \frac{e^t}{2} \left( 1 - \frac{e^t}{2} \right)^{-1}$$

$$= \frac{e^t}{2} \frac{e^t}{2 - e^t}$$

$$\Rightarrow M_X(t) = \frac{e^t}{2 - e^t}$$

$$(ii) E(X) = \left[ \frac{d}{dt} [M_X(t)] \right]_{t=0}$$

$$= \left[ \frac{d}{dt} \left( \frac{e^t}{2 - e^t} \right) \right]_{t=0}$$

$$= \left[ \frac{(2 - e^t)e^t - e^t(0 - e^t)}{(2 - e^t)^2} \right]_{t=0}$$

$$= \left[ \frac{2e^t - e^{2t} + e^{2t}}{(2 - e^t)^2} \right]_{t=0} \because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= \left[ \frac{2e^t}{(2 - e^t)^2} \right]_{t=0} = \frac{2}{1}$$

$$\Rightarrow E(X) = 2$$

$$E(X^2) = \left[ \frac{d^2}{dt^2} [M_X(t)] \right]_{t=0}$$

$$= \left[ \frac{d}{dt} \left( \frac{2e^t}{(2-e^t)^2} \right) \right]_{t=0}$$

$$= \left[ \frac{(2-e^t)^2 2e^t - 2e^t 2(2-e^t)(-e^t)}{(2-e^t)^4} \right]_{t=0} = \frac{2+4}{1} = 6$$

(iii) Variance =  $E(X^2) - [E(X)]^2 = 6 - 4$

$\text{Var}(X) = 2$

**Problems under MGF of discrete random variable**

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

If  $X$  is a continuous RV and if  $M_X(t)$  is known, then  $\mu'_r$

$$= r! \times \text{coeff of } t^r \text{ in } M_X(t)$$

1. If a random variable “X” has the MGF,  $M_X(t) = \frac{2}{2-t}$ , find the variance of X.

**Solution:**

Given  $M_X(t) = \frac{2}{2-t} = 2(2-t)^{-1}$

$$M_X'(t) = -2(2-t)^{-2}(-1)$$

$$= 2(2-t)^{-2}$$

$$M_X'(t=0) = 2(2-0)^{-2} = \frac{2}{4} = \frac{1}{2}$$

$$M_X''(t) = -4(2-t)^{-3}(-1)$$

$$= 4(2 - t)^{-3}$$

$$M_x''(t = 0) = 4(2 - 0)^{-3} = \frac{4}{8} = \frac{1}{2}$$

$$\text{Var}(X) = E(X^2) - E[(X)]^2$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

**2. Let X be a RV with PDF  $f(x) = ke^{-2x}, x \geq 0$ . Find (i)  $k$ , (ii) MGF, (iii) Mean and (iv) variance**

**Solution:**

Given  $f(x) = ke^{-2x}; 0 \leq x < \infty$

**(i) To find  $k$**

$$\Rightarrow \int_0^{\infty} f(x) dx = 1$$

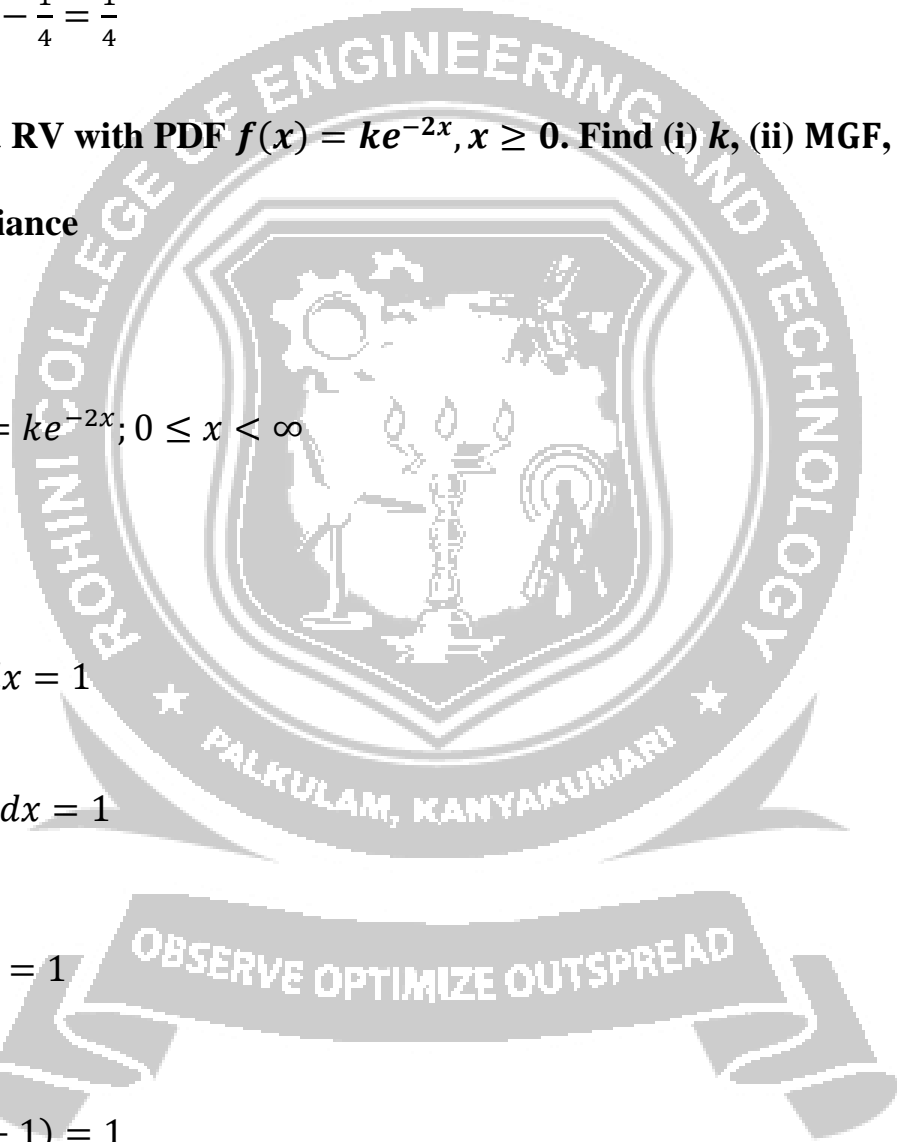
$$\Rightarrow \int_0^{\infty} ke^{-2x} dx = 1$$

$$\Rightarrow k \left[ \frac{e^{-2x}}{-2} \right]_0^{\infty} = 1$$

$$\Rightarrow \frac{k}{-2} (e^{-\infty} - 1) = 1$$

$$\Rightarrow \frac{k}{-2} (0 - 1) = 1$$

$$\Rightarrow \frac{k}{2} = 1$$





$$\Rightarrow k = 2$$

(ii)  $M_X(t) = E[e^{tx}]$

$$= \int_0^{\infty} e^{tx} f(x) dx$$

$$= 2 \int_0^{\infty} e^{tx} e^{-2x} dx = 2 \int_0^{\infty} e^{tx-2x} dx$$

$$= 2 \int_0^{\infty} e^{-(2-t)x} dx = 2 \left[ \frac{e^{-(2-t)x}}{-(2-t)} \right]_0^{\infty} = 2 \left( 0 + \frac{1}{2-t} \right)$$

$$M_X(t) = \frac{2}{2-t}$$

(iii) To find Mean and Variance

$$M_X(t) = \frac{2}{2 \left( 1 - \frac{t}{2} \right)} = \left( 1 - \frac{t}{2} \right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \dots$$

Coefficient of  $t = \frac{1}{2}$  Coefficient of  $t^2 = \frac{1}{2^2}$

Mean  $E(X) = \mu'_1 = 1! \times \text{coefficient of } t \Rightarrow E(X) = \frac{1}{2}$

$$E(X^2) = 2! \times \text{coefficient of } t^2 = 2 \times \frac{1}{2^2} = \frac{1}{2}$$

(iv) Variance  $= E(X^2) - [E(X)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4}$

$$\text{Var}(X) = \frac{1}{4}$$

3. Let  $X$  be a continuous RV with PDF  $f(x) = Ae^{-\frac{x}{3}}$ ;  $x \geq 0$ . Find (i)  $A$ , (ii)

MGF, (iii) Mean and (iv) variance

**Solution:**

Given  $f(x) = Ae^{-\frac{x}{3}}; 0 \leq x \leq \infty$

**(i) To find A**

$$\Rightarrow \int_0^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^{\infty} Ae^{-\frac{x}{3}} dx = 1$$

$$\Rightarrow A \left[ \frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_0^{\infty} = 1$$

$$\Rightarrow -3A(0 - 1) = 1$$

$$\Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}$$

$$\therefore f(x) = \frac{1}{3} e^{-\frac{x}{3}}; 0 \leq x \leq \infty$$

$$(ii) M_X(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x) dx = \frac{1}{3} \int_0^{\infty} e^{tx} e^{-\frac{x}{3}} dx$$

$$= \frac{1}{3} \int_0^{\infty} e^{tx - \frac{x}{3}} dx = \frac{1}{3} \int_0^{\infty} e^{-(\frac{1}{3} - t)x} dx = \frac{1}{3} \left[ \frac{e^{-(\frac{1}{3} - t)x}}{-(\frac{1}{3} - t)} \right]_0^{\infty}$$

$$= \frac{1}{3} \left[ 0 + \frac{1}{\frac{1}{3} - t} \right] = \frac{1}{3} \frac{1}{\frac{1}{3} - t}$$

$$= (1 - 3t)^{-1}$$

**(iii) To find mean and variance:**

$$M_X(t) = (1 - 3t)^{-1}$$

$$= 1 + 3t + 9t^2 + 27t^3 + \dots$$

coefficient of  $t = 3$

coefficient of  $t^2 = 9$

$$E(X) = 1! \times \text{coefficient of } t \text{ in } M_X(t) = 1 \times 3$$

Mean = 3

$$E(X^2) = 2! \times \text{coefficient of } t^2 \text{ in } M_X(t)$$

$$= 2 \times 9 = 18$$

$$(iv) \text{ Variance} = E(X^2) - [E(X)]^2 = 18 - 9$$

$$\text{Var}(X) = 9$$

**4. Let  $X$  be a continuous random variable with the pdf**

$$f(x) = \begin{cases} x & ; 0 < x < 1 \\ 2 - x & ; 1 < x < 2 \\ 0 & ; \text{elsewhere} \end{cases} \text{ Find (i) MGF, (ii) Mean and variance.}$$

**Solution:**

Since  $X$  is defined in the region  $0 < x < 2$ ,  $X$  is a continuous RV.

$$M_X(t) = E[e^{tX}] = \int_0^2 e^{tx} f(x) dx$$

$$= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx} (2 - x) dx$$

$$\begin{aligned}
 &= \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx \\
 &= \left[ x \left( \frac{e^{tx}}{t} \right) - 1 \left( \frac{e^{tx}}{t^2} \right) \right]_0^1 + \left[ (2-x) \frac{e^{tx}}{t} - (-1) \frac{e^{tx}}{t^2} \right]_1^2 \\
 &= \left[ 1 \left( \frac{e^t}{t} \right) - 1 \left( \frac{e^t}{t^2} \right) - \left( \frac{-1}{t^2} \right) \right] + \left[ 0 + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2} \right] \\
 &= \left[ \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2} \right] = \frac{1}{t^2} - \frac{2e^t}{t^2} + \frac{e^{2t}}{t^2}
 \end{aligned}$$

$$M_X(t) = \frac{1 - 2e^t + e^{2t}}{t^2}$$

**To find Mean and Variance:**

$$\begin{aligned}
 M_X(t) &= \frac{1}{t^2} \left[ 1 - 2 \left( 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) \right. \\
 &\quad \left. + \left( 1 + \frac{2t}{1!} + \frac{2^2 t^2}{2!} + \frac{2^3 t^3}{3!} + \frac{2^4 t^4}{4!} + \dots \right) \right]
 \end{aligned}$$

$$\mu'_r = r! \times \text{coefficient of } t^r$$

$$\text{Coefficient of } t = -\frac{2}{3!} + \frac{2^3}{3!} = \frac{-2}{6} + \frac{8}{6} = 1$$

$$\text{Coefficient of } t^2 = -\frac{2}{4!} + \frac{2^4}{4!} = \frac{14}{24} = \frac{7}{12}$$

$$\mu'_1 = 1! \times \text{coefficient of } t$$

$$\mu'_1 = 1$$

$$\text{Mean} = 1$$

$$\mu'_2 = 2! \times \text{coefficient of } t^2; \mu'_2 = 2 \times \frac{7}{12} = \frac{7}{6}$$

$$\text{Variance} = \mu'_2 - (\mu'_1)^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

**5. Let  $X$  be a continuous random variable with PDF  $f(x) = \frac{1}{2a}; -a < x < a$ . Then find the M.G.F of  $X$ .**

**Solution:**

Let  $X$  is a continuous random variable defined in  $-a < x < a$ .

$$\begin{aligned} M_x(t) &= E[e^{tx}] \\ &= \int_{-a}^a e^{tx} f(x) dx \\ &= \int_{-a}^a e^{tx} \frac{1}{2a} dx \\ &= \frac{1}{2a} \left( \frac{e^{tx}}{t} \right)^a e^x - e^{-x} = 2 \sin hx \\ &= \frac{1}{2a} \left( \frac{e^{ta} - e^{-ta}}{t} \right) \\ &= \frac{1}{2a} \frac{2 \sinh at}{t} \end{aligned}$$

$$M_x(t) = \frac{\sinh at}{at}$$