5.3 $Runge - Kutta\ Method$

$k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$k_4 = hf(x_n + h, y_n + k_3)$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$y_1 = y_0 + \Delta y$$

1. $Using\ Runge - Kutta\ Method\ \ find\ y\ at\ x = 0.1$

$$if\ \frac{dy}{dx} = x + y^2\ \ with\ y(0) = 1$$

$solution$:

$Given\ y' = f(x, y) = x + y^2\ and$

$$x_0 = 0\ and\ y_0 = 1$$
$$x_1 = 0.1\ and\ y_1 = ?$$
$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$

$By\ Runge - Kutta\ Method$

$k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$k_4 = hf(x_n + h, y_n + k_3)$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$k_1 = hf(x_0, y_0)$

$= (0.1)f(0,1)$

$= (0.1)[0 + 1^2] = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= (0.1)f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$

$= (0.1)f(0.05, 1.05)$

$= (0.1)[0.05 + (1.05)^2] = 0.1[0.05 + 1.1025]$

$= 0.1[1.1525]$

$= 0.1153$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.1)f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1153}{2}\right)$

$= (0.1)f(0.05, 1.0577)$

$= (0.1)[0.05 + (1.0577)^2] = 0.1[0.05 + 1.1187]$

$= 0.1[1.1687]$

$= 0.1169$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= hf(0 + 0.1, 1 + 0.1169)$

$= hf(0.1, 1.1169)$

$= (0.1)[0.1 + (1.1169)^2] = 0.1[0.1 + 1.2475]$

$= 0.1[1.3475]$

$= 0.1348$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.1 + 2(0.1153) + 2(0.1169) + 0.1348]$

$= \frac{1}{6}[0.4686] = 0.0781$

$$y_1 = y_0 + \Delta y = 1 + 0.0781 = 1.0781$$

$2.\ Using\ Runge-Kutta\ Method\ \ find\ y\ at\ x=0.2$

$$if\ \frac{dy}{dx}=\frac{y^2-x^2}{y^2+x^2}\ \ with\ y(0)=1$$

solution:

$$Given\ y'=f(x,y)=\frac{y^2-x^2}{y^2+x^2}\ and$$

$$x_0=0\ and\ y_0=1$$

$$x_1=0.2\ and\ y_1=?$$

$$h=\ x_1-\ x_0=0.2-0=0.2$$

$By\ Runge-Kutta\ Method$

$$k_1=hf(x_n,y_n)$$

$$k_2=hf\left(x_n+\frac{h}{2},y_n+\frac{k_1}{2}\right)$$

$$k_3=hf\left(x_n+\frac{h}{2},y_n+\frac{k_2}{2}\right)$$

$$k_4=hf(x_n+h,y_n+k_3)$$

$$\Delta y=\tfrac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$k_1=hf(x_0,y_0)$$

$$=(0.2)f(0,1)$$

$$=(0.2)\left[\frac{1^2-0^2}{1^2+0^2}\right]=(0.2)\left[\frac{1}{1}\right]=0.2$$

$$k_2=hf\left(x_0+\frac{h}{2},y_0+\frac{k_1}{2}\right)$$

$$=(0.2)f\left(0+\frac{0.2}{2},1+\frac{0.2}{2}\right)$$

$$=(0.2)f(0.1,1.1)$$

$$=(0.2)\left[\frac{1.1^2-0.1^2}{1.1^2+0.1^2}\right]=0.2\left[\frac{1.2}{1.22}\right]$$

$= 0.2[0.9836]$

$= 0.1967$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.2)f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1967}{2}\right)$

$= (0.2)f(0.1, 1.0984)$

$= (0.2)\left[\frac{1.0984^2 - 0.1^2}{1.0984^2 + 0.1^2}\right] = 0.1967$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= h\,f(0 + 0.2, 1 + 0.1967)$

$= (0.2)f(0.2, 1.1967)$

$= (0.2)\left[\frac{1.1967^2 - 0.2^2}{1.1967^2 + 0.2^2}\right] = 0.1891$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.2 + 2(0.1967) + 2(0.1967) + 0.1891]$

$= 0.19598$

$y_1 = y_0 + \Delta y$

$= 1 + 0.19598 = 1.19598$

3. Apply the fourth order Rungekutta method to find an approximate value of y when x=0.2 and x=0.4 given that $y' = x + y, y(0) = 1 \;\; with\, h = 0.2$

Solution :

Given $y' = x + y, y(0) = 1 \; with\, h = 0.2$

$$x_0 = 0, y_0 = 1$$

$$f(x,y)=x+y$$

$By\ Runge-Kutta\ Method$

$k_1=hf(x_n,y_n)$

$k_2=hf\left(x_n+\frac{h}{2},y_n+\frac{k_1}{2}\right)$

$k_3=hf\left(x_n+\frac{h}{2},y_n+\frac{k_2}{2}\right)$

$k_4=hf(x_n+h,y_n+k_3)$

$\Delta y=\frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$k_1=hf(x_0,y_0)$

$=0.2(x_0+y_0)$

$=0.2(0+1)$

$=0.2$

$k_2=hf\left(x_0+\frac{h}{2},y_0+\frac{k_1}{2}\right)$

$=0.2\ f\left(0+\frac{0.2}{2},1+\frac{1}{2}(0.2)\right)$

$=0.2\ f(0.1,1.1)$

$=0.2(0.1+1.1)$

$=0.2(1.2)$

$=0.24$

$k_3=hf\left(x_0+\frac{h}{2},y_0+\frac{k_2}{2}\right)$

$=0.2\ f\left(0+\frac{0.2}{2},1+\frac{1}{2}(0.24)\right)$

$=0.2\ f(0.1,1.12)$

$=0.2(0.1+1.12)$

$=0.2(1.22)$

=0.244

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.2\, f(0 + 0.2, 1 + 0.244)$

$= 0.2\, f(0.2, 1.244)$

$= 0.2\,(0.2 + 1.244)$

$= 0.2\,(1.444)$

$= 0.2888$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.2 + 2(0.24) + 2(0.244) + 0.2888]$

=0.2428

$y_1 = y(0.2) = y_0 + \Delta y$

= 1+0.2428

=1.2428

Now starting from (x_1, y_1) $we\ get(x_1, y_1)$

Here $x_1 = 0.2, y_1 = 1.242$

$k_1 = hf(x_1, y_1)$
$= 0.2(x_1 + y_1)$

$= 0.2(0.2 + 1.2428)$

$= 0.28856$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$= 0.2\, f\left(0.2 + \frac{0.2}{2}, 1.2428 + \frac{1}{2}(0.28856)\right)$

$= 0.2\, f(0.3, 1.38708)$

$= 0.2\,(0.3 + 1.38708)$

$= 0.337416$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= 0.2\, f\left(0.2 + \frac{0.2}{2}, 1.2428 + \frac{1}{2}(0.337416\right)$

$= 0.2\, f(0.3, 1.411508)$

$= 0.2\,(0.3 + 1.411508)$

$k_3 = 0.3423$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= 0.2\, f(0.2 + 0.2, 1.2428 + 0.3423)$

$= 0.2\, f(0.4, 1.5851)$

$= 0.2\,(0.4 + 1.5851)$

$= 0.39702$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$\Delta y = \frac{1}{6}[0.28856 + 2(0.337416) + 2(0.3423) + 0.39702]$

$= \frac{1}{6}[0.28856 + 0.674832 + 0.6846 + 0.39702]$

$\Delta y = 0.3408$

$$y(0.2) = y_2 = y_1 + \Delta y$$

$$= 1.2428 + 0.3408$$

$$y(0.2) = 1.5836$$

4. Apply the fourth order Rungekutta method to find an approximate value of y when x=0.2 and x=0.4 given that $y'' + xy' + y = 0,$

$$y(0) = 1 \;\; y'(0) = 0 \; with \; h = 0.2$$

Solution :

$$y'' = -xy' - y, \qquad y(0) = 1 \;\; y'(0) = 0 \; with \; h = 0.2$$

$$x_0 = 0, y_0 = 1$$

Setting y'=z

The equation becomes

$y'' = z' = -xz - y,$

$\frac{dy}{dx} = z = f_1(x, y, z)$

$\frac{dz}{dx} = -xz - y = f_2(x, y, z)$

By Algorithm,

$k_1 = hf_1(x_0, y_0, z_0) = (0.1)f_1(0,1,0) = (0.1)(0) = 0$

$l_1 = hf_2(x_0, y_0, z_0) = (0.1)f_2(0,1,0) = (0.1)(-1) = -0.1$

$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_2 = 0.1f_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1)f_1(0.05,1, -0.05) = (0.1)(-0.05) = -0.005$

$l_2 = (0.1)f_2(0.05,1 - 0.05) = (0.1)[1 + (0.05)(0.05) - 1]$

$= 0.09975$

$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1)f_1(0.05,0.9975, -0.0499) = (0.1)(-0.0499) = -0.00499$

$l_3 = (0.1)f_2(0.05,0.9975, -0.0499)$

$= (0.1)(0.05 + 0.9975 - 0.0499)$

=-0.09950

$k_3 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.1)f_1(0.1,0.99511, -0.0995) = (0.1)(-0.0995) = -0.00995$

$l_4 = (0.1)f_2(0.1,0.99511, -0.0995)$

$= (0.1)(-(0.1 - 0.0995 + 0.99511)$

=-0.0985

$y_1 = y(0.1) = y_0 + \Delta y$=$1 + \frac{1}{6}[0 + 2(-0.005) + 2(-0.00499) - 0.00995]$

=0.9950