UNIT-V

NUMERICAL SOLUTION OF ORDIARY DIFFERENTIAL EQUATIONS

5.1 Taylor's series

$$y_1 = y_0 + \frac{h}{1!}y_0{}' + \frac{h^2}{2!}y_0{}'' + \frac{h^3}{3!}y_0{}''' + \cdots \ldots ..$$

1. Using Taylor's series method find y at $x = 0.1$ if $\frac{dy}{dx} = x^2y - 1, y(0) = 1.$

Solution

Given, $y' = x^2y - 1$ and $x_0 = 0, y_0 = 1, h = 0.1$

Taylor's series formula for y_1 is,

$$y_1 = y_0 + \frac{h}{1!}y_0{}' + \frac{h^2}{2!}y_0{}'' + \frac{h^3}{3!}y_0{}''' + \cdots \ldots ..$$

$y' = x^2y - 1$ $\qquad y_0{}' = x_0{}^2y_0 - 1 = 0 - 1 = -1$

$$y'' = 2xy + x^2y' y_0{}'' = 2x_0y_0 + x_0{}^2y_0{}' = 0 + 0 = 0$$

$$y''' = 2(xy' + y) + x^2y'' + 2xy' y_0{}''' = 4x_0y_0{}' + 2y_0 + x_0{}^2y_0{}''$$

$$= 2xy' + 2y + x^2y'' + 2xy' = 4(0)(-1) + 2(1) + (0)^2(0) = 2$$

$= 4xy' + 2y + x^2y''$

$$y_1 = y_0 + \frac{h}{1!}y_0{}' + \frac{h^2}{2!}y_0{}'' + \frac{h^3}{3!}y_0{}''' + \cdots \ldots ..$$

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \cdots \ldots ..$$

$$= 1 - 0.1 + + \frac{(0.1)^3}{3} + \cdots \ldots . = 0.9003$$

2. Solve, $y' = x + y; y(0) = 1,$ by Taylor's series method. Find the values y at $x = 0.1$ $and\ x = 0.2.$

$Solution$:

Given, $y' = x + y$ and $x_0 = 0, y_0 = 1, h = 0.1$

Taylor's series formula for y_1 is,

$$y_1 = y_0 + \frac{h}{1!}y_0{}' + \frac{h^2}{2!}y_0{}'' + \frac{h^3}{3!}y_0{}''' + \cdots \ldots ..$$

$y' = x + y$	$y_0{}' = x + y_0 = 0 + 1 = 1$
$y'' = 1 + y'$	$y_0{}'' = 1 + y_0{}' = 1 + 1 = 2$
$y''' = y''$	$y_0{}''' = y_0{}'' = 2$
$y^{iv} = y'''$	$y_0{}^{iv} = y_0{}''' = 2$

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(1) + \frac{0.1^2}{2!}(2) + \frac{0.1^3}{3!}(2) + \cdots \ldots ..$$

$$y(0.1) = 1 + 0.1 + 0.1^2 + \frac{0.1^3}{3} + \cdots \ldots . = 1.1103$$

Take, $x_1 = 0.1, y_1 = 1.1103, h = 0.1$

$$y_2 = y_1 + \frac{h}{1!}y_1{'} + \frac{h^2}{2!}y_1{''} + \frac{h^3}{3!}y_1{'''} + \cdots \ldots ..$$

$y_1{'} = x_1 + y_1 = 0.1 + 1.1103 = 1.2103$
$y_1{''} = 1 + y_1{'} = 1 + 1.2103 = 2.2103$
$y_1{'''} = y_1{''} = 2.2103$

$$y_2 = y(0.2) = 1.1103 + \frac{0.1}{1!}(1.2103) + \frac{0.1^2}{2!}(2.2103) + \frac{0.1^3}{3!}(2.2103) + \cdots \ldots ..$$

$$= 1.1103 + 0.12103 + 0.01105 = 1.2427$$

3. By means of Taylor's series expansion, find y at $x = 0.1$ *and* $x = 0.2$ correct to three significant digits given $\frac{dy}{dx} - 2y = 3e^x, y(0) = 0$

Solution

$$x_0 = 0, y_0 = 0, h = 0.1, x_1 = 0.1, x_2 = 0.2$$

$y' = 2y + 3e^x$	$y_0{'} = 2y_0 + 3e^{x_0} = 3$
$y'' = 2y' + 3e^x$	$y_0{''} = 2y_0{'} + 3e^{x_0} = 9$
$y''' = 2y'' + 3e^x$	$y_0{'''} = 2y_0{''} + 3e^{x_0} = 21$

Taylor's series formula for y_1 is,

$$y_1 = y_0 + \frac{h}{1!}y_0{'} + \frac{h^2}{2!}y_0{''} + \frac{h^3}{3!}y_0{'''} + \cdots \ldots ..$$

$$y_1 = y(0.1) = 0 + \frac{0.1}{1!}(3) + \frac{0.1^2}{2!}(9) + \frac{0.1^3}{3!}(21) + \cdots \ldots ..$$

$$= 0.3 + 0.045 + 0.0035 = 0.3486 = 0.349$$

$y_1{'} = 2y_1 + 3e^{x_1} = 4.012887$
$y_1{''} = 2y_1{'} + 3e^{x_1} = 11.025774$

$$\boxed{\begin{aligned} y_1''' &= 2y_1'' + 3e^{x_1} \\ &= 25.367 \end{aligned}}$$

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \ldots ..$$

$$y_2 = y(0.2) = 0.3486 + \frac{0.1}{1!}(4.012887) + \frac{0.1^2}{2!}(11.025774) + \frac{0.1^3}{3!}(25.99808) + \cdots \ldots .$$

$$= 0.8110156 = 0.811.$$

$y(0.1) = 0.349$

$y(0.2) = 0.811$

4. *Using Taylors Serious Method find y at* $x = 0.1$

if $\frac{dy}{dx} = x^2 - y$ *with* $y(0) = 1$

Given $y' = x^2 - y$ *and*

$$x_0 = 0 \text{ and } y_0 = 1$$
$$x_1 = 0.1 \text{ and } y_1 = ?$$
$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$

$y' = x^2 - y$	$y_0' = x_0{}^2 - y_0 = 0 - 1 = -1$
$y'' = 2x - y'$	$y_0'' = 2x_0 - y_0' = 0 - (-1) = 1$
$y''' = 2 - y''$	$y_0''' = 2 - y_0'' = 2 - (1) = 1$

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..$$

$$y_1 = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..$$

$$= 1 - 0.1 + \frac{0.01}{2} + \frac{0.001}{6}$$

$$= 1 - 0.1 + 0.005 + 0.00016$$

$$= 0.90516$$

5. *Using Taylors Serious Method find y at* $x = 0.2$ *if* $\frac{dy}{dx} = 1 - 2xy$

with $y(0) = 0$

Solution

iven $y' = 1 - 2xy$ *and*

$$x_0 = 0 \text{ and } y_0 = 0$$
$$x_1 = 0.2 \text{ and } y_1 = ?$$
$$h = x_1 - x_0 = 0.2 - 0 = 0.2$$

$y' = 1 - 2xy$	$y_0' = 1 - 2x_0y_0 = 1 - 0 = 1$
$y'' = -2(xy' + y)$	$y_0'' = -2(x_0y_0' - y_0) = -2(0 - 0) = 0$
$y''' = -2(xy'' + y' + y')$	$y_0''' = -2\left(y_0' - x_0y_0'' + y_0'\right) = -2(1 - (0) + 1)$ $= -2(2) = -4$

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..$$

$$y_1 = 0 + \frac{0.2}{1}(1) + \frac{(0.2)^2}{2}(0) + \frac{(0.2)^3}{6}(-4) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..$$

$$= 0.2 + 0 + \frac{0.008}{6}(-4)$$

$$= 0.2 - 0.00533 = 0.19475$$