ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MATHEMATICS

LECTURE NOTES ON

BA4201 / QUANTITATIVE TECHNIQUES FOR DECISION MAKING

UNIT V : QUEUING MODELS

QUEUEING MODELS

Model: I : $(M \,/\, M \,/\, 1) : (\infty/FIFO)$

1**. In a railway marshalling yard goods trains arrive at a rate of 30 trains per day. Assume that the inter arrival time follows exponential distribution and the service time distribution is also exponential with an average of 36 minutes. Calculate the following**

(i) The mean queue size.

(ii) The probability that the queue size is atleast 10.

(iii) If the input of the trains increases to an average of 33 per day, what will be the change in the above quantities.

Solution:

Mean arrival rate: $\boldsymbol{\lambda}$

⇒1day – 30 arrivals

24 hours – 30 arrivals

1 hour – $\frac{30}{24}$ arrivals

60 min – $\frac{30}{24}$ arrivals

1 min – $\frac{30}{24*60}$ arrivals

1 min – $\frac{1}{48}$ arrivals

⇒λ =1/48 per min

Mean service rate : $\boldsymbol{\mu}$

36 mins – 1 service

1 min – 1/36 service

⇒ μ = $1/_{36}$ per min

To find $\boldsymbol{\rho = \frac{\lambda}{\mu} = \frac{1/_{48}}{1/_{36}} = \frac{3}{4}}$

(i)Mean line length:

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{({}^1/_{48})^2}{{}^1/_{36}\,({}^1/_{36} - {}^1/_{48})}$$

$$= \frac{0.000434}{0.0277(0.0277-0.0208)} = 2.27 \text{ trains}$$

(ii)**P(queue size exceeds 10)** $= P(N > 10)$

$$P(N > K) = \rho^{K+1}$$

$$\Rightarrow P(N > 10) = \rho^{10+1} = \left(\frac{\lambda}{\mu}\right)^{11} = \left(\frac{3}{4}\right)^{11} = 0.0422$$

(iii)If the input of the train increases to an average of 33 per day

Mean arrival rate: $\boldsymbol{\lambda}$

⇒1day – 33 arrivals

24 hours – 33 arrivals

1 hour – $\frac{33}{24}$ arrivals

60 min – $\frac{33}{24}$ arrivals

1 min – $\frac{33}{24*60}$ arrivals

1 min – $\frac{11}{480}$ arrivals

$\Rightarrow \lambda = \frac{11}{480}$ / min

To find $\rho = \frac{\lambda}{\mu} = \frac{{}^{11}/_{480}}{{}^1/_{36}} = \frac{33}{40}$

(i)**Mean line length:**

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{({}^{11}/_{480})^2}{{}^1/_{36}\,({}^1/_{36} - {}^{11}/_{480})}$$

$L_q = 3.889$ trains

Change in mean line length $= 3.889 - 2.27 = 1.619$

(ii)P(queue size exceeds 10) $= P(N > 10)$

$$P(N > K) = \rho^{K+1}$$

$$\Rightarrow P(N > 10) = \rho^{10+1}$$

$$\Rightarrow P(N > 10) = \rho^{10+1} = \left(\frac{33}{40}\right)^{11} = 0.1205$$

Change in queue size $= 0.1205 - 0.0422 = 0.0783$

2. **Customers arrive at a watch repair shop according to a Poisson process at a rate of one per every 10 minutes and the service time is an exponential random variable with mean 8 minutes. Find (i) the average number of customers in the shop (ii) the average waiting time of a customer spends in the shop (iii) the average time a customer spends in waiting for service (iv) utilization factor.**

Solution:

Mean arrival rate: $\boldsymbol{\lambda}$

10 min – 1 arrivals

1 min – $\frac{1}{10}$ arrivals

⇒λ =1/10 per min

Mean service rate : $\boldsymbol{\mu}$

8 mins – 1 service

1 min – $1/8$ service

⇒ μ = $1/8$ per min

To find $\boldsymbol{\rho}$:

$$\Rightarrow \rho = \frac{\lambda}{\mu} = \frac{1/10}{1/8} = \frac{4}{5}$$

(i)Average number of customers in the shop

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/10}{1/8 - 1/10}$$

$L_s = 4$ customers

(ii)Average waiting time a customer spends in the shop.

$$W_s = \frac{L_s}{\lambda} \ (or) \ = \frac{1}{\mu - \lambda}$$

$$W_s = \frac{4}{1/10} = 40 \text{ minutes}$$

(iii)Average time a customer spends in waiting for service W_q

$$W_q = \frac{L_q}{\lambda} \ (or) \ \frac{\lambda}{\mu(\mu - \lambda)}$$

$$W_q = \frac{1/10}{1/8(1/8 - 1/10)} = \frac{1/10}{1/320} = 32$$

$$W_q = 32 \text{ minutes}$$

(iv)utilization factor $\rho = \frac{\lambda}{\mu}$

$$\Rightarrow \rho = \frac{4}{5}$$

3. **In a store there is only one cashier at its counter. Nine customers arrive on an average 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service rate, find**

(i) The average number of customers in the system.

(ii) The average number of customers in queue or average queue length.

(iii) The average time a customer waits before being served.

(iv)The average time a customer spends in the system

(v) The traffic intensity

Solution:

Mean arrival rate: $\boldsymbol{\lambda}$

5 min – 9 arrivals

1 min – $\frac{9}{5}$ arrivals

$\Rightarrow \lambda = 9/5$ per min

Mean service rate : $\boldsymbol{\mu}$

5 mins – 10 service

1 min – ${}^{10}/_{5}$ service

$\Rightarrow \mu = 2/$ min

(i)Average number of customers in the system L_s

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{{}^{9}/_{5}}{2-{}^{9}/_{5}}$$

$$L_s = 9 \text{ customers}$$

(ii)average number of customers in the queue L_q

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{({}^{9}/_{5})^2}{2(2-{}^{9}/_{5})} = 8.1$$

$$L_q = 8 \text{ customers}$$

(iii)Average time a customer waits before being served W_q

$$W_q = \frac{L_q}{\lambda}$$

$$W_q = \frac{8.1}{{}^{9}/_{5}} = 4.5 \text{ minutes}$$

(iv)Average time a customer spends in the system W_s

$$W_s = \frac{L_s}{\lambda} = \frac{9}{{}^{9}/_{5}} = 5 \text{ minutes}$$

(v) Traffic intensity ρ

$$\rho = \frac{\lambda}{\mu} = \frac{{}^{9}/_{5}}{2} = {}^{9}/_{10}$$

4. **Customers arrive at the first class ticket counter of a theatre at a rate of 12 per hour. There is 1 clerk servicing the customers at the rate of 30 per hour.**

(i) What is the probability that there is no customer at the counter?

(ii) What is the probability that there are more than 2 customers at the counter?

(iii) What is the probability that there is no customer waiting to be served?

(iv) What is the probability that a customer is being served and nobody is waiting?

Solution:

Mean arrival rate: $\boldsymbol{\lambda}$

⇒ 1 hour – 12 arrivals

60 min – 12 arrivals

1 min – $\frac{12}{60}$ arrivals

1 min – $\frac{1}{5}$ arrivals

⇒λ =1/5 per min

Mean service rate : $\boldsymbol{\mu}$

⇒ 1 hour – 30 service

60 mins – 30 services

1 min – $30/60$ service

⇒ $\mu = 1/2$ per min

(i)Prob that there is no customer at the counter P_0

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{1/5}{1/2} = 3/5$$

To find $\boldsymbol{\rho}$:

$$\rho = \frac{\lambda}{\mu} = \frac{1/5}{1/2} = 2/5$$

(ii)Prob that there are more than 2 customers at the counter

$$P(N > 2) = 1 - P(N \leq 2) = 1 - (P_0 + P_1 + P_2)$$

$$P_n = (1 - \rho)\rho^n$$

$$P_1 = (1-\rho)\rho^1 = \left(1 - {}^{2}/_{5}\right){}^{2}/_{5} = {}^{6}/_{25}$$

$$P_2 = (1-\rho)\rho^2 = \left(1 - {}^{2}/_{5}\right)({}^{2}/_{5})^2 = {}^{12}/_{125}$$

$$P(N > 2) = 1 - \left({}^{3}/_{5} + {}^{6}/_{25_1} + {}^{12}/_{125}\right) = 1 - {}^{117}/_{125}$$

$$= {}^{8}/_{125} = 0.064$$

(iii) Prob that there is no customer waiting to be serve

= Prob that atmost one customer at the counter

$$P(N \leq 1) = P_0 + P_1 = {}^{3}/_{5} + {}^{6}/_{25} = {}^{21}/_{25} = 0.84$$

(iv) Prob that a customer is being served and nobody is waiting = Prob exactly one customer at the counter getting the service

$$P_1 = {}^{6}/_{25} = 0.24$$

$$\Rightarrow P_1 = 0.24$$

5. **A T.V. repairman finds that the time spend on his job has an exponential distribution with mean 30 minutes. If he repair sets in the order in which they came in and if the arrival of sets is approximately Poisson with an average rate of 10 per 8 – hour day,**

(i)What is the repairman's expected idle time each day?

(ii) What is the average queue length?

(iii)Find the average number of jobs in the system.

Solution:

Mean arrival rate: λ

$\Rightarrow$ 8 hours – 10 arrivals

1 hour – $\frac{10}{8}$ arrivals

60 mins – $\frac{10}{8}$ arrivals

1 min – $\frac{10}{8*60}$ arrivals

1 min – $\frac{1}{48}$ arrivals

$\Rightarrow \lambda = {}^{1}/_{48}$ per min

Mean service rate : $\boldsymbol{\mu}$

30 mins – 1 services

1 min – ${}^{1}/_{30}$ service

$\Rightarrow \mu = {}^{1}/_{30}$ per min

To find $\boldsymbol{\rho}$:

$$\rho = \frac{\lambda}{\mu} = \frac{{}^{1}/_{48}}{{}^{1}/_{30}} = {}^{5}/_{8}$$

(i)Idle time = system empty

$$= P_0 = 1 - \rho = 1 - {}^{5}/_{8} = {}^{3}/_{8}$$

$\therefore$ *Expected idle time per day* $= 8 * {}^{3}/_{8} = 3$ hours

(ii)Average queue length L_q

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{({}^{1}/_{48})^2}{{}^{1}/_{30}\,({}^{1}/_{30} - {}^{1}/_{48})} = {}^{25}/_{24} = 1.04\ TV$$

(iii)Average no. of jobs in the system L_s

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{{}^{1}/_{48}}{{}^{1}/_{30} - {}^{1}/_{48}} = {}^{5}/_{3} = 1.667$$

Model: II : $\boldsymbol{M / M / s : \infty / FCFS\ (FIFO)}$

(Infinite capacity, Multi server Queueing model)

1. **A supermarket has 2 girls running up sales at the counters. If the service time for each customer is exponential with mean 4 minutes, and if people arrive in a Poisson fashion at the rate of 10 per hour**

(i) What is the probability of having to wait for service?

(ii) What is the expected percentage of idle time for each girl?

(iii) If a customer has to wait, what is the expected length of waiting time?

Solution:

Given $s = 2$

Mean arrival rate: λ

1 hour – 10 arrivals

60 mins – 10 arrivals

1 min – $\frac{10}{60}$ arrivals

1 min – $\frac{1}{6}$ arrivals

$\Rightarrow \lambda = 1/6$ per min

Mean service rate : μ

4 min – 1 service

1 min – $1/4$ service

$\Rightarrow \mu = 1/4$ per min

(i)To find ρ:

$$\rho = \frac{\lambda}{s\mu} = \frac{1/6}{2*1/4} = 1/3$$

Let $\frac{\lambda}{\mu} = \frac{1/6}{1/4} = 2/3$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{1-\rho}\right]^{-1}$$

$$= \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{1/6}{1/4}\right)^n + \frac{1}{2!}\left(\frac{1/6}{1/4}\right)^2 \frac{1}{(1-1/3)}\right]^{-1}$$

$$= \left[\frac{1}{0!}\left(\frac{2}{3}\right)^0 + \frac{1}{1!}\left(\frac{2}{3}\right)^1 + \frac{1}{2\left(\frac{2}{3}\right)}\left(\frac{2}{3}\right)^2\right]^{-1}$$

$$= 1 + \left(\frac{2}{3}\right) + 0.33 = 1.996$$

(ii)Prob. of a customer has to wait for the service

$$P[N \geq s] = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{1-\rho} \cdot P_0$$

$$= \frac{1}{2!}\left(\frac{2}{3}\right)^2 \frac{1}{1-{}^1/_3} * 1.996$$

$$= 0.222 * 2.994$$

$$P[N \geq s] = 0.6658$$

(iii)Average queue length L_q

$$L_q = \frac{1}{s.s!}\left(\frac{\lambda}{\mu}\right)^{s+1} \frac{1}{(1-\rho)^2} P_0$$

$$= \frac{1}{2*2!}\left(\frac{2}{3}\right)^{2+1} \frac{1}{\left(1-{}^1/_3\right)^2} * 1.996$$

$$= 0.111 * 4.4991$$

$$L_q = 0.4994$$

(iv)Average time spend by a customer in the queue W_q

$$W_q = \frac{L_q}{\lambda} = \frac{0.4994}{{}^1/_6}$$

$$W_q = 2.9964\ mins.$$

(v)Idle time $= 1 - \rho$

$$= 1 - {}^1/_3 = {}^2/_3 = 0.6667$$

% of idle time for each girl $= 0.6667 * 100 = 67\%$

2. **There are 3 typists in an office. Each typist can type an average of 6 letters per hour. If letters for being typed at the rate of 15 letters per hour.**

(i) What fraction of time all the typist will be busy?

(ii) What is the average number of letters waiting to be typed?

(iii) What is the average time a letter has to spend waiting and for being typed?

(iv) What is the probability that a letter will take longer than 20 minutes. waiting typed and being typed.

Solution:

Mean arrival rate: $\boldsymbol{\lambda}$

1 hour – 15 arrivals

60 mins – 15 arrivals

1 min – $\frac{15}{60}$ arrivals

1 min – $\frac{1}{4}$ arrivals

$\Rightarrow \lambda = {}^{1}/_{4}$ per min

Mean service rate : $\boldsymbol{\mu}$

1 hour – 6 service

60 mins – 6 service

1 min – ${}^{6}/_{60}$ service

1 min – ${}^{1}/_{10}$ service

$\Rightarrow \mu = {}^{1}/_{10}$ per min

i)To find ρ: $s = 3$

Let $\frac{\lambda}{\mu} = \frac{{}^{1}/_{4}}{{}^{1}/_{10}} = {}^{5}/_{2}$

$$\rho = \frac{\lambda}{s\mu} = \frac{{}^{1}/_{4}}{3 * {}^{1}/_{10}} = {}^{5}/_{6}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{1-\rho}\right]^{-1}$$

$$= \left[\sum_{n=0}^{3-1} \frac{1}{n!}\left(\frac{5}{2}\right)^n + \frac{1}{3!}\left(\frac{5}{2}\right)^3 \frac{1}{\left(1 - {}^{5}/_{6}\right)}\right]^{-1}$$

$$= \left[\frac{1}{0!}\left(\frac{5}{2}\right)^0 + \frac{1}{1!}\left(\frac{5}{2}\right)^1 + \frac{1}{2!}\left(\frac{5}{2}\right)^2 + \frac{1}{1}(15.625)\right]^{-1}$$

$$= \left[1 + \frac{5}{2} + \frac{25}{8} + 15.625\right]^{-1} = 0.0449$$

$$P_0 = 0.0449$$

(i)P(all the typists are busy) = P(arrival has to wait)

$$P[N \geq s] = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{1-\rho} \,.\, P_0$$

$$= \frac{1}{3!}\left(\frac{5}{2}\right)^3 \frac{1}{1-{}^5/_6} * 0.0449$$

$$= 2.6041 * 0.2694$$

$$P[N \geq s] = 0.7015$$

(ii)Average no. of letters waiting to be typed

$$L_q = \frac{1}{s.\,s!}\left(\frac{\lambda}{\mu}\right)^{s+1} \frac{1}{(1-\rho)^2} \; P_0$$

$$= \frac{1}{3*3!}\left(\frac{5}{2}\right)^{3+1} \frac{1}{\left(1-{}^5/_6\right)^2} * 0.0449$$

$$= 2.1701 * 1.6164 = 3.50$$

$$L_q = 3.50$$

(iii)Average time a letter has to spend for waiting and for being typed

$$W_s = \frac{L_s}{\lambda}$$

$$L_s = L_q + \frac{\lambda}{\mu} = 3.50 + {}^5/_2 = 6$$

$$W_s = \frac{6}{{}^1/_4} = 24 \text{ minutes}$$

(iv)P(that a letter will take longer than 20 minutes waiting to be typed and being typed) , $t = 20$ minutes

$$P[W_s > t] = e^{-\mu t}\left[1 + \frac{\left(\frac{\lambda}{\mu}\right)^s\left[1-e^{-\mu t\left(s-1-\frac{\lambda}{\mu}\right)}\right]P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)\left(s-1-\frac{\lambda}{\mu}\right)}\right]$$

$$= e^{-1/_{10}*20}\left[1 + \frac{\left(\frac{5}{2}\right)^s\left[1-e^{-\mu t\left(s-1-\frac{\lambda}{\mu}\right)}\right]P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)\left(s-1-\frac{\lambda}{\mu}\right)}\right]$$

$$= 0.1353\left[1 + 15.625 \times \frac{1 - 2.7182}{-0.5} \times 0.0449\right]$$

$$= 0.4614$$

3. **Four counters are being run on the frontier of a country to check the passports and necessary papers of the tourists. The tourist chooses a counter a random. If the arrival at the frontier is Poisson at the rate of $\boldsymbol{\lambda}$, and the service time is exponential $\frac{\boldsymbol{\lambda}}{\mathbf{2}}$, what is the steady – state average queue at each counter.**

Solution:

Given, $s = 3 + 1 = 4$

Mean arrival rate: $\boldsymbol{\lambda}$

$$\lambda = 1$$

Mean service rate : $\boldsymbol{\mu}$

$$\mu = 1/_2$$

To find ρ: $s = 4$

Let $\frac{\lambda}{\mu} = \frac{1}{1/_2} = 2$

$$\rho = \frac{\lambda}{s\mu} = \frac{1}{4*1/_2} = 1/_2$$

$$P_0 = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s\frac{1}{1-\rho}\right]^{-1}$$

$$= \left[\sum_{n=0}^{4-1} \frac{1}{n!}(2)^n + \frac{1}{4!}(2)^4 \frac{1}{(1 - {}^{1}/_{2})}\right]^{-1}$$

$$= \left[\frac{1}{0!}(2)^0 + \frac{1}{1!}(2)^1 + \frac{1}{2!}(2)^2 + \frac{1}{3!}(2)^3 + \frac{1}{4!* {}^{1}/_{2}}(2)^4\right]^{-1}$$

$$= \left[1 + 2 + \frac{4}{2} + \frac{8}{6} + \left(\frac{2}{24} * 16\right)\right]^{-1}$$

$$= \left[\frac{23}{3}\right]^{-1} = 0.1304$$

$$P_0 = 0.1304$$

(i)Average queue at each counter L_q

$$L_q = \frac{1}{s.s!}\left(\frac{\lambda}{\mu}\right)^{s+1} \frac{1}{(1-\rho)^2} P_0$$

$$= \frac{1}{4* 4!}(2)^{4+1} \frac{1}{(1-{}^{1}/_{2})^2} * 0.1304$$

$$= 0.3333 * 0.5216$$

$$L_q = 0.1738$$

4. **A telephone exchange has two long distance operators. The telephone company finds that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes (i) what is the probability that a subscriber will have to wait for his long distance call during peak hours of the day. (ii) If the subscribers will wait and are served in turn, what is the expected waiting time.**

Solution:

Mean arrival rate: λ

1 hour – 15 arrivals

60 mins – 15 arrivals

1 min – $\frac{15}{60}$ arrivals

1 min – $\frac{1}{4}$ arrivals

$\Rightarrow \lambda = {}^{1}/_{4}$ per min

Mean service rate : μ

5 mins – 1 service

1 min – ${}^{1}/_{5}$ service

$\Rightarrow \mu = {}^{1}/_{5}$ per min

(i)To find ρ: $s = 2$

Let $\frac{\lambda}{\mu} = \frac{1/4}{1/5} = {}^{5}/_{4}$

$\rho = \frac{\lambda}{s\mu} = \frac{1/4}{2*1/5} = {}^{5}/_{8}$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{1-\rho}\right]^{-1}$$

$$= \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{5}{4}\right)^n + \frac{1}{2!}\left(\frac{5}{4}\right)^2 \frac{1}{(1-5/8)}\right]^{-1}$$

$$= \left[\frac{1}{0!}\left(\frac{5}{4}\right)^0 + \frac{1}{1!}\left(\frac{5}{4}\right)^1 + \frac{1}{2!*3/8}\left(\frac{5}{4}\right)^2\right]^{-1}$$

$$= \left[1 + \frac{5}{4} + (1.3333 * 1.5625)\right]^{-1} = 0.2320$$

$P_0 = 0.2320$

(i)P(that an arrival has to wait)

$$P[N \geq s] = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{1-\rho} \, . \, P_0$$

$$= \frac{1}{2!}\left(\frac{5}{4}\right)^2 \frac{1}{1-5/8} * 0.2320$$

$$= 0.5 * 1.5625 * 0.61866 = 0.48$$

$P[N \geq s] = 0.48\, mins$

(ii)Expected waiting time including service W_s

$$W_s = \frac{L_s}{\lambda}, L_s = L_q + \frac{\lambda}{\mu}$$

$$L_q = \frac{1}{s.s!}\left(\frac{\lambda}{\mu}\right)^{s+1} \frac{1}{(1-\rho)^2} \; P_0$$

$$= \frac{1}{2 * \; 2!}\left(\frac{5}{4}\right)^{2+1} \frac{1}{\left(1-{}^{5}/_{8}\right)^2} * 0.2320$$

$$= \frac{1}{4} * 1.9531 * 1.6497$$

$$L_q = 0.8054$$

$$L_s = 0.8054 + \frac{5}{4} = 2.0554$$

$$W_s = \frac{2.0554}{{}^{1}/_{4}} = 8.2216 \text{ minutes}$$

$$W_s = 8.2216 \text{ minutes}$$

Model: III : $M/M/1 : k/FIFO$

1. **The local one – person barber shop accommodate a maximum of 5 people at a time (4 waiting and 1 getting hair cut). Customers arrive according to a Poisson distribution with mean 5/hr. The barber cuts hair at an average rate of 4/hr (exponential service time).**

(i) What percentage of time is the barber idle?

(ii) What fraction of the potential customers are turned away?

(iii) What is the expected number of customers waiting for a hair – cut?

(iv) How much time can a customer expect to spend in the barber shop?

Solution:

Given $k = 5$

Mean arrival rate: λ

1 hour – 5 arrivals

60 mins – 5 arrivals

1 min – $\frac{5}{60}$ arrivals

1 min – $\frac{1}{12}$ arrivals

$\Rightarrow \lambda = {}^{1}/_{12}$ per min

Mean service rate : $\boldsymbol{\mu}$

1 hour – 4service

60 mins – 4 service

1 min – ${}^{4}/_{60}$ service

1 min – ${}^{1}/_{15}$ service

$\Rightarrow \mu = {}^{1}/_{15}$ per min

To find $\boldsymbol{\rho}$:

$$\rho = \frac{\lambda}{\mu} = \frac{{}^{1}/_{12}}{{}^{1}/_{15}} = {}^{5}/_{4}$$

(i)% of time the barber idle= system empty= P_0

$$P_0 = \frac{1-\rho}{1-\rho^{k+1}}, \lambda \neq \mu$$

$$= \frac{1-{}^{5}/_{4}}{1-({}^{5}/_{4})^{5+1}} = 0.0888$$

% of idle time= $0.0888 * 100 = 8\%$

(ii)P(a potential customer turned away) $P_k = \rho^k * P_0$

$$= \left({}^{5}/_{4}\right)^5 * 0.0888 = 0.2709$$

(iii)Expected no. of customers waiting for a hair cut L_q

$$L_q = L_s - \frac{\lambda'}{\mu} \ \ where\ \lambda' = \mu(1-P_0)$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$$

$$L_s = \frac{5/4}{1-5/4} - \frac{6*(5/4)^6}{1-(5/4)^6} = 3.13\ customers$$

To find λ':

$$\lambda' = \mu(1 - P_0)$$

$$= 1/15\,(1 - 0.0888) = 0.06074$$

$$L_q = 3.13 - \frac{0.06074}{1/15} = 2.218 \text{ customers}$$

(iv)Customer spend in the barber shop W_s

$$W_s = \frac{L_s}{\lambda'} = \frac{3.13}{0.06074} = 51.53\ minutes.$$

2. **At a railway station, only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait, while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6/hr and the railway station can handle them on an average of 6/hr. Assuming Poisson arrivals and exponential service distribution.**

(i) Find the probabilities for the numbers of trains in the system.

(ii) Find the average waiting time of a new train coming into the yard.

(iii) If the handling rate is doubled then what about (i) and (ii).

Solution:

Mean arrival rate: λ

1 hour – 6 arrivals

60 mins – 6 arrivals

1 min – $\frac{6}{60}$ arrivals

1 min – $\frac{1}{10}$ arrivals

$\Rightarrow \lambda = 1/10$ per min

Mean service rate : μ

1 hour – 6 services

60 mins – 6 services

1 min – $6/_{60}$ service

1 min – $1/_{10}$ service

⇒ $\mu = 1/_{10}$ per min

To find ρ:

$$\rho = \frac{\lambda}{\mu} = \frac{1/_{10}}{1/_{10}} = 1, \quad k = 2 + 1 = 3$$

(i)P(the number of trains in the system)

$$P_n = \frac{1}{k+1} \qquad if \quad \lambda = \mu$$

$$= \frac{1}{3+1} = \frac{1}{4} = 0.25$$

$$P_n = 0.25$$

(ii)average time a new train coming into the yard W_s

$$W_s = \frac{L_s}{\lambda'} \ \ where\ \lambda' = \mu(1 - P_0)$$

$$L_s = \frac{k}{2}, \ \ \lambda = \mu$$

$$L_s = \frac{3}{2} = 1.5\ trains$$

To find λ'

$$\lambda' = \mu(1 - P_0)$$

$$\lambda' = 1/_{10}\,(1 - 0.25) = 0.075$$

$$W_s = \frac{1.5}{0.075} = 20\ mins$$

If the rate of handling is doubled

Mean service rate : μ

1 hour – 12 services

60 mins – 12 services

1 min – $^{12}/_{60}$ service

1 min – $^{1}/_{5}$ service

$\Rightarrow \mu = {}^{1}/_{5}$ per min, $\lambda \neq \mu$

(i)$P_0 = \frac{(1-\rho)\rho^0}{1-\rho^{k+1}} \quad if \;\; \lambda \neq \mu \qquad ; \;\; \rho = \frac{\lambda}{\mu} = \frac{^1/_{10}}{^1/_5} = {}^1/_2$

$= \frac{(1-{}^1/_2)({}^1/_2)^0}{1-({}^1/_2)^{3+1}} = {}^8/_{15} = 0.5333$

(ii) $W_s = \frac{L_s}{\lambda'} \;\; where\; \lambda' = \mu(1-P_0)$

$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$

$L_s = \frac{^1/_2}{1-{}^1/_2} - \frac{4*({}^1/_2)^4}{1-({}^1/_2)^4} = 0.7334$ trains

To find λ'

$\lambda' = \mu(1-P_0)$

$= {}^1/_5\,(1-0.5333) = 0.0933$

$W_s = \frac{L_s}{\lambda'} = \frac{0.7334}{0.0933} = 7.86$

$W_s = 7.86\; minutes.$

3. **Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour the waiting room does not accommodate more than 14 patients. Diagnosis time per patient is exponential with mean rate of 20 per hour. Find out (i)The effective arrival rate at the clinic. (ii) The probability that an arriving patient will not have to wait. (iii) the expected waiting time until a patient is discharged from the clinic.**

Solution:

Mean arrival rate:λ

1 hour – 30 arrivals

60 mins – 30 arrivals

1 min – $\frac{30}{60}$ arrivals

1 min – $\frac{1}{2}$ arrivals

$\Rightarrow \lambda = {}^{1}/_{2}$ per min

Mean service rate : μ

1 hour – 20 services

60 mins – 20 services

1 min – ${}^{20}/_{60}$ service

1 min – ${}^{1}/_{3}$ service

$\Rightarrow \mu = {}^{1}/_{3}$ per min

To find ρ:

$$\rho = \frac{\lambda}{\mu} = \frac{{}^{1}/_{2}}{{}^{1}/_{3}} = {}^{3}/_{2}$$

Given, $k = 14 + 1$ (14 waiting and 1 in the doctor room)

(i)Effective arrival time $\lambda' = \mu(1 - P_0), \lambda \neq \mu$

$$P_0 = \frac{(1-\rho)}{1-\rho^{k+1}} = \frac{(1-{}^{3}/_{2})}{1-({}^{3}/_{2})^{15+1}} = 0.00076$$

$$\lambda' = {}^{1}/_{3}(1 - 0.00076) = 0.33308\, per\ minutes.$$

(ii)P(that an arriving patient will not wait) P_0

$$P_0 = 0.00076$$

(iii)Expected waiting time W_s

$$W_s = \frac{L_s}{\lambda'} \ \ where\ \lambda' = \mu(1 - P_0)$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} \;, \; \lambda \neq \mu$$

$$L_s = \frac{3/2}{1-3/2} - \frac{15*(3/2)^{15+1}}{1-(3/2)^{15+1}} = 12.022 \; customers$$

$$L_s = 12.022 \; customers$$

$$W_s = \frac{L_s}{\lambda'} = \frac{12.022}{0.33308} = 36.09 \; mins$$

4. **A car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10/hr. The length of time each car spends in the car park is exponential distribution with mean of 2hrs. how many cars are in the car park on average? Probability of a newly arriving customer finding the car park full and leaving to park his car elsewhere.**

Solution:

Mean arrival rate: $\boldsymbol{\lambda}$

1 hour – 10 arrivals

60 mins – 10 arrivals

1 min – $\frac{10}{60}$ arrivals

1 min – $\frac{1}{6}$ arrivals

⇒$\lambda = 1/6$ per min

Mean service rate : $\boldsymbol{\mu}$

2 mins – 1 services

1 min – $1/2$ service

⇒ $\mu = 1/2$ per min

To find $\boldsymbol{\rho}$:

$$\rho = \frac{\lambda}{\mu} = \frac{1/6}{1/2} = 1/3$$

Given, $k = 5$

(i)Average no. of cars in the car park L_s

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} \ , \lambda \neq \mu$$

$$L_s = \frac{1/3}{1-1/3} - \frac{6*(1/3)^{5+1}}{1-(1/3)^{5+1}} = 0.4972$$

(ii)P(of a newly arriving customer finding the car park full and leaving to park his car elsewhere) P_n

$$P_n = \rho^n * P_0, \lambda \neq \mu$$

$$P_0 = \frac{(1-\rho)}{1-\rho^{k+1}} = \frac{(1-1/3)}{1-(1/3)^6} = 0.6675$$

$$P_5 = \rho^5 * 0.6675 = (1/3)^5 * 0.6675 = 0.00274$$

5. **In a railway marshalling yard, goods train arrive at a rate of 15 trains per day. Assume that the inter arrival time follows an exponential distribution and the service time distribution is also exponential with an average of 33 minutes. Calculate**

(i) The probability that the yard is empty.

(ii)Average number of trains in the system.

Assuming that the line capacity of the yard is 9 trains.

Solution:

Mean arrival rate: λ

15 mins – 1 arrival

1 min – $\frac{1}{15}$ arrival

$\Rightarrow \lambda = 1/15$ per min

Mean service rate : μ

33 mins – 1 service

1 min – $1/33$ service

$\Rightarrow \mu = {}^{1}/_{33}$ per min

To find ρ:

$$\rho = \frac{\lambda}{\mu} = \frac{{}^{1}/_{15}}{{}^{1}/_{33}} = {}^{11}/_{5}$$

Given, $k = 4$

(i)P(the yard is empty) P_0

$$P_0 = \frac{(1-\rho)}{1-\rho^{k+1}} = \frac{(1-{}^{11}/_{5})}{1-({}^{11}/_{5})^5} = 0.02374$$

(ii)average no. of trains in the system L_s

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} \;, \lambda \neq \mu$$

$$L_s = \frac{{}^{11}/_{5}}{1-{}^{11}/_{5}} - \frac{5*({}^{11}/_{5})^5}{1-({}^{11}/_{5})^5} = 3.26\, train$$

Model: IV : $M/M/s : k/FIFO$

1. A 2 – person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when all 5 chairs are full, leave without entering barber – shop. Customers arrive at an average rate of 4/hr and spend an average of 12 minutes in the barber's chair. Compute $P_0,\ P_1,\ P_7,\ E(N_q),\ E(W_s)$.

Solution:

Given, 5 waiting and 2 in barber's chair.

Mean arrival rate: λ

$s = 2,\ k = 7$

1 hour – 4 arrivals

60 mins – 4 arrivals

1 min – $\frac{4}{60}$ arrivals

1 min – $\frac{1}{15}$ arrivals

$\Rightarrow \lambda = {}^{1}/_{15}$ per min

Mean service rate : μ

12 mins – 1 services

1 min – ${}^{1}/_{12}$ service

$\Rightarrow \mu = {}^{1}/_{12}$ per min

To find ρ:

$$\rho = \frac{\lambda}{\mu s} = \frac{{}^{1}/_{15}}{2 * {}^{1}/_{12}} = {}^{2}/_{5}$$

$$\frac{\lambda}{\mu} = \frac{{}^{1}/_{15}}{{}^{1}/_{12}} = {}^{4}/_{5}$$

(i) $P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k} \left(\frac{\lambda}{s\mu}\right)^{n-s}\right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{2-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \sum_{n=2}^{7} \left(\frac{\lambda}{s\mu}\right)^{n-2}\right]^{-1}$$

$$= \left\{\frac{1}{0!}\left(\frac{4}{5}\right)^0 + \frac{1}{1!}\left(\frac{4}{5}\right)^1 + \frac{1}{2!}\left(\frac{4}{5}\right)^2 \left[\left(\frac{2}{5}\right)^0 + \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5\right]\right\}^{-1}$$

$$= \left\{1 + \frac{4}{5} + \frac{8}{25}\left[1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \frac{32}{3125}\right]\right\}^{-1}$$

$$P_0 = 0.4289$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0,\ n \le s, \qquad n = 1,\ s = 2$$

(ii) $P_1 = \frac{1}{1!}\left(\frac{4}{5}\right)^1 * 0.4289 = 0.34312$

$$P_n = \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0 \qquad for \quad s < n \le k$$

(iii) $P_7 = \frac{1}{2!\,2^{7-2}}\left(\frac{4}{5}\right)^{n7} * 0.4289 = 0.00140, \qquad 2 < n \leq k = 7$

(iv) $W_s = \frac{L_s}{\lambda'}$ where $\lambda' = \mu[s - \sum_{n=0}^{s-1}(s-n)\,P_n]$

To find λ'

$$\lambda' = {}^{1}\!/_{12}\,[2 - \sum_{n=0}^{2-1}(2-n)\,P_n]$$

$$= {}^{1}\!/_{12}\,[2 - (2-0)P_0 - (2-1)P_1]$$

$$= {}^{1}\!/_{12}\,[2 - 2 * 0.4289 - 1 * 0.34312]$$

$$\lambda' = 0.0666$$

$$L_q = \frac{\rho}{s!\,(1-\rho)^2}\left(\frac{\lambda}{\mu}\right)^s [1 - \rho^{k-s} - (k-s)(1-\rho)\rho^{k-s}]P_0$$

$$= \frac{\left({}^{2}\!/_{5}\right)}{s!\,\left(1 - {}^{2}\!/_{5}\right)^2}\left(\frac{4}{5}\right)^s \left[1 - \left({}^{2}\!/_{5}\right)^{7-2} - (7-2)\left(1 - {}^{2}\!/_{5}\right)\left({}^{2}\!/_{5}\right)^{7-2}\right] * 0.4$$

$$= \frac{16}{45} * (0.98976 - 0.03072) * 0.4289$$

$$L_q = 0.1462\ customers$$

$$L_s = L_q + \frac{\lambda'}{\mu} = 0.1462 + \frac{0.0666}{{}^{1}\!/_{12}}$$

$$L_s = 0.9454\ customers.$$

$$W_s = \frac{L_s}{\lambda'} = \frac{0.9454}{0.0666} = 14.195\ minutes$$

$$W_s = 14.195\ minutes.$$