

## Poisson Process

Let  $X(t)$  denotes the number of occurrences of a certain event in the interval  $(0, t)$ . Then the discrete random process  $\{X(t)\}$  is called the Poisson process if it satisfies the following postulates.

### Postulates of Poisson process

- (i)  $P[1 \text{ occurrence in } (t, t + \Delta t)] = \lambda\Delta t + O(\Delta t) = P_1(\Delta t)$
- (ii)  $P[0 \text{ occurrence in } (t, t + \Delta t)] = 1 - \lambda\Delta t + O(\Delta t) = P_0(\Delta t)$
- (iii)  $P[2 \text{ or more occurrences in } (t, t + \Delta t)] = O(\Delta t)$

### Some Applications of Poisson Process:

- (i) Arrival of customers in simple queuing system
- (ii) The number of wrong telephone calls at a switch board.
- (iii) The number of passengers entering a railway station on a given day.
- (iv) The emission of radioactive particles.

### Probability law of Poisson process:

Let  $\lambda$  be the number of customers per unit time.  $P_m(t)$  be the probability of  $m$  occurrences, of the event in the interval  $(0, t)$ .

$P_0(t)$  denotes the probability of 0 occurrences, in the interval  $(0, t)$ .

Probability of zero occurrence of the event in the time interval 0 to  $t + \Delta t$  is given by  $P_0(t + \Delta t)$ .

Also  $P_0(t + \Delta t)$  can be rewritten as

$P_0(t + \Delta t)$  = Probability of zero occurrence in the interval  $(0, t)$  and also in the interval  $(t, t + \Delta t)$

$$\text{i.e., } P_0(t + \Delta t) = P_0(1 - \lambda\Delta t)$$

$$= P_0(t) - \lambda\Delta t P_0(t)$$

$$\Rightarrow P_0(t + \Delta t) - P_0(t) = -\lambda\Delta t P_0(t)$$

$$\Rightarrow \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t)$$

Taking lim as  $\Delta t \rightarrow 0$

$$\Rightarrow \lim_{n \rightarrow 0} \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$$\Rightarrow \frac{d}{dt} [P_0(t)] = -\lambda P_0(t)$$

$$\Rightarrow \frac{d[P_0(t)]}{P_0(t)} = -\lambda dt$$

Integrating on both sides

$$\Rightarrow \int \frac{d[P_0(t)]}{P_0(t)} = \int -\lambda dt$$

$$\Rightarrow \log P_0(t) = -\lambda t + c$$

$$\Rightarrow P_0(t) = e^{-\lambda t} A \text{ where } A = e^c \quad \dots (1)$$

To find A, Put  $t = 0$

$$\Rightarrow P_0(0) = A$$

$$\Rightarrow A = 1$$

$$(1) \Rightarrow P_0(t) = e^{-\lambda t}$$

Now  $P_m(t) = P[X(t) = m]$

$$= P(m \text{ occurrences of the event in } (0, t))$$

$$P_m(t + \Delta t) = P[X(t + \Delta t) = m]$$

$$= P(m \text{ occurrences of the event in } (0, t + \Delta t))$$

$$= P(m-1 \text{ occurrences in } (0, t) \text{ and } 1 \text{ occurrence in } (t, t + \Delta t)) +$$

$P(m \text{ occurrences of the event in } (0, t) \text{ and } 0 \text{ occurrence in } (t, t + \Delta t))$

$$= P_{m-1}(t)\lambda\Delta t + P_m(t)(1 - \lambda\Delta t)$$

$$= P_{m-1}(t)\lambda\Delta t + P_m(t) - \lambda P_m(t)\Delta t$$

$$\Rightarrow P_m(t + \Delta t) - P_m(t) = \lambda\Delta t[P_{m-1}(t) - P_m(t)]$$

$$\Rightarrow \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = \lambda[P_{m-1}(t) - P_m(t)]$$

Taking lim as  $\Delta t \rightarrow 0$

$$\lim_{\Delta t \rightarrow 0} \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = \lambda[P_{m-1}(t) - P_m(t)]$$

$$\Rightarrow \frac{d}{dt} [P_m(t)] = \lambda[P_{m-1}(t) - P_m(t)]$$

$$\Rightarrow \frac{d}{dt} P_m(t) + \lambda P_m(t) = \lambda P_{m-1}(t)$$

$$\Rightarrow P_m(t)e^{\lambda t} = \int \lambda P_{m-1}(t) e^{\lambda t} dt$$

Put  $m = 1$

$$\Rightarrow P_1(t)e^{\lambda t} = \int \lambda P_0(t) e^{\lambda t} dt$$

$$= \int \lambda e^{-\lambda t} e^{\lambda t} dt$$

$$= \int \lambda dt$$

$$\Rightarrow P_1(t)e^{\lambda t} = \lambda t$$

$$\Rightarrow P_1(t)e^{\lambda t} = \frac{(\lambda t)^1 e^{-\lambda t}}{1!}$$

This is the first order Poisson probability distribution.

$$\text{In general } P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

### Properties of Poisson Process:

#### Property:1

**Poisson Process is a Markov process.**

#### Proof:

Let us take the conditional probability distribution of  $X(t_3)$  given the past values of  $X(t_2)$  and  $X(t_1)$ .

Assume that  $t_3 > t_2 > t_1$  and  $n_3 > n_2 > n_1$

Consider  $P[X(t_3) = n_3 / X(t_2) = n_2, X(t_1) = n_1]$

$$\begin{aligned}
&= \frac{P[X(t_3) = n_3, X(t_2) = n_2, X(t_1) = n_1]}{P[X(t_1) = n_1, X(t_2) = n_2]} \\
&= \frac{e^{-\lambda(t_3-t_2)} \cdot \lambda^{n_3-n_2} \cdot (t_3-t_2)^{n_3-n_2}}{(n_2-n_3)!} \\
&= P[X(t_3) = n_3 / X(t_2) = n_2]
\end{aligned}$$

Hence the conditional probability distribution  $X(t_3)$  given that values of the process  $X(t_2)$  and  $X(t_1)$  depends only on the most recent value  $X(t_2)$  of the process.

Hence Poisson process is a Markov process.

Hence the proof.

### Property: 2

#### Additive Property

**The sum of two independent Poisson processes is a Poisson Process.**

#### Proof:

Let  $X_1(t)$  and  $X_2(t)$  be two independent Poisson process with parameter  $\lambda_1 t$  and  $\lambda_2 t$  respectively.

Let  $X(t) = X_1(t) + X_2(t)$

Now  $P[X(t) = n] = P[(X_1(t) + X_2(t)) = n]$

$$\Rightarrow \sum_{r=0}^n P[X_1(t) = r]P[X_2(t) = n - r]$$

$$\Rightarrow \sum_{r=0}^n \frac{e^{-\lambda_1 t} (\lambda_1 t)^r}{r!} \frac{e^{-\lambda_2 t} (\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= \frac{e^{-(\lambda_1 + \lambda_2)t}}{n!} \sum_{r=0}^n \frac{n!}{r!(n-r)!} (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= \frac{e^{-(\lambda_1 + \lambda_2)t}}{n!} \sum_{r=0}^n n C_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= \frac{e^{-(\lambda_1 + \lambda_2)t}}{n!} (\lambda_1 t + \lambda_2 t)^n$$

Which is a Poisson process with parameter  $\lambda_1 + \lambda_2$ .

Hence the proof.

### Property: 3

**Difference of two independent Poisson processes not a Poisson Process.**

### Proof:

Let  $X_1(t)$  and  $X_2(t)$  be two independent Poisson processes with parameter  $\lambda_1 t$  and  $\lambda_2 t$  respectively.

$$\text{Let } X(t) = X_1(t) - X_2(t)$$

$$\text{Now } E[X(t)] = E[X_1(t)] - E[X_2(t)]$$

$$= \lambda_1 t - \lambda_2 t$$

$$\begin{aligned}
\text{Now, } E[X^2(t)] &= E[(X_1(t) - X_2(t))^2] \\
&= E[X_1^2 t + X_2^2 t - 2X_1(t)X_2(t)] \\
&= E[X_1^2 t] + E[X_2^2 t] - 2E[X_1(t)X_2(t)] \\
&= \lambda_1^2 t^2 + \lambda_1 t + \lambda_2^2 t^2 + \lambda_2 t - 2E[X_1(t)]E[X_2(t)] \\
&= \lambda_1^2 t^2 + \lambda_2^2 t^2 + \lambda_1 t + \lambda_2 t - 2\lambda_1 \lambda_2 t \\
&= (\lambda_1 + \lambda_2)t + (\lambda_1^2 + \lambda_2^2)t^2 - 2\lambda_1 \lambda_2 t^2 \\
&= (\lambda_1 + \lambda_2)t + (\lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2)t^2 \\
&= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2 \\
&\neq (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2
\end{aligned}$$

Hence  $X(t) = X_1(t) + X_2(t)$  is not a Poisson process.

Hence the proof.

#### Property: 4

#### Distribution of the interval arrival time

The inter arrival time of a Poisson process with parameter  $\lambda$  follows exponential distribution with mean  $\frac{1}{\lambda}$ .

#### Proof:

Let the two consecutive events be  $E_i$  and  $E_{i+1}$ .

Let  $E_i$  take place at time  $t_i$  and  $E_{i+1}$  take place at time  $t_i + T$ .

Let  $T$  be the interval between the occurrences  $E_i$  and  $E_{i+1}$  then  $T$  is a continuous random variable.

Now,  $P(T > t) = P(t < T)$

$$\begin{aligned}
 &= P(E_{i+1} \text{ does not occur in } (t_i, t_i + T)) \\
 &= P(\text{no event occurs in the interval of length } t) \\
 &= P[X(t) = 0] \\
 &= \frac{e^{-\lambda t} (\lambda t)^0}{0!} \\
 &= e^{-\lambda t}
 \end{aligned}$$

The cumulative distribution of  $T$  is

$$\begin{aligned}
 F(t) &= P(T \leq t) = 1 - P(T > t) \\
 &= 1 - e^{-\lambda t}
 \end{aligned}$$

Hence the probability density function is

$$\begin{aligned}
 f(t) &= \frac{d}{dt} [F(t)] \\
 &= -e^{-\lambda t} (-\lambda)
 \end{aligned}$$



$$= \lambda e^{-\lambda t}$$

Which is the probability density function of exponential distribution with mean

$$\frac{1}{\lambda}.$$

Hence the proof.

### Property: 5

If the number of occurrences of an event E in an interval of length t is a Poisson process  $X(t)$  with parameter  $\lambda t$  and if such occurrences of E has a constant probability of being recorded and the recording are independent of each other. Then the number  $N(t)$  of the recorded occurrences in time t is also a Poisson process.

### Proof:

$$P[N(t) = n] = \sum_{r=0}^n [E \text{ occurs } (n+r) \text{ times and } n \text{ of them are recorded}]$$

$$= \sum_{r=0}^n \frac{e^{-\lambda t} (\lambda t)^r}{(n+r)!} (n+r) C_n p^n q^{n+r-n}$$

$$= e^{-\lambda t} \sum_{r=0}^n \frac{(\lambda t)^n (\lambda t)^r}{(n+r)!} \frac{(n+r)!}{n!(n+r-n)!} p^n q^r$$

$$= \frac{e^{-\lambda t} (\lambda t)^n}{n!} p^n \sum_{r=0}^n \frac{(\lambda t q)^r}{r!}$$

$$= \frac{e^{-\lambda t}(\lambda t p)^n}{n!} \left[ 1 + \frac{\lambda t p}{1!} + \frac{(\lambda t p)^2}{2!} + \dots \right]$$

$$= \frac{e^{-\lambda t}(\lambda t p)^n}{n!} e^{\lambda t q}$$

$$= \frac{e^{-\lambda t(1-q)}(\lambda t p)^n}{n!}$$

$$= \frac{e^{-\lambda t p}(\lambda t p)^n}{n!}$$

= Probability density function of Poisson process

Hence  $N(t)$  is a Poisson process.

Hence the proof.

### Problems under Probability law of Poisson Process:

**1. Suppose the customer arrive at a bank according to a Poisson Process with mean rate of 3 per minute. Find the probability that during a time interval of two minutes. (i) Exactly 4 customer arrive. (ii) Greater than 4 customer arrive. (iii) Fewer than 4 customers arrive.**

### Solution:

Let  $X(t)$  denotes the number of customers arrived during the interval  $(0, t)$ .

Then  $\{X(t)\}$  follows Poisson process.

Given :  $\lambda = 3/\text{min}$  and  $t = 2 \text{ min}$ .

$$P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2,3 \dots \dots \dots$$

$$i. e. ., P(X(2) = n) = \frac{e^{-6}(6)^n}{n!}, n = 0,1,2,3 \dots \dots$$

(i)P(Exactly 4 customer arrive during a time interval of two minutes)

$$P(X(2) = 4) = \frac{e^{-6}(6)^4}{4!} = \frac{3.2124}{24} = 0.13385$$

$$P(X(2) = 4) = 0.13385$$

(ii)P( greater than 4 customer arrive during a time interval of two minutes) =

$$P(X(2) > 4) = 1 - P(X(2) \leq 4)$$

$$= 1 - \{P(X(2) = 0) + P(X(2) = 1) + P(X(2) = 2) +$$

$$P(X(2) = 3) + P(X(2) = 4)\}$$

$$= 1 - \left[ \frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!} + \frac{e^{-6}(6)^4}{4!} \right]$$

$$= 1 - e^{-6} \left[ 1 + 6 + \frac{36}{2} + 36 + 54 \right]$$

$$= 1 - e^{-6}[115] = 1 - 0.285 = 0.715$$

$$P(X(2) > 4) = 0.715$$

(iii)P(less than 4 customer arrive during a time interval of two minutes)

$$P(X(2) < 4) = P(X(2) = 0) + P(X(2) = 1) + P(X(2) = 2) + P(X(2) = 3)$$

$$= \left[ \frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!} \right]$$

$$= e^{-6}[1 + 6 + 8 + 36]$$

$$= e^{-6}[61] = 0.1512$$

$$P(X(2) < 4) = 0.1512$$

**2. A hard disk fails in a computer system it follows a Poisson distribution with mean rate of 1 per week. Find the probability, that 2 weeks have elapsed since last failure. If we have 5 extra hard disks and the next supply is not due in 10 weeks. Find the probability that the machine will not be out of order in the next 10 weeks.**

**Solution:**

Given  $\lambda = 1$

$$P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0, 1, 2, 3 \dots$$

To find the probability that 2 weeks have elapsed since last failure.

$$P(X(2) = 0) = \frac{e^{-1 \times 2}(1 \times 2)^0}{0!}$$

$$= e^{-2}$$

$$= 0.1353$$

Number of extra hard disk = 5

The probability that the machine will not be the probability that the machine will not be out of order in the next weeks is  $P(X(10) \leq 5)$

$$= \{P(X(10) = 0) + P(X(10) = 1) + P(X(10) = 2) + P(X(10) = 3) + P(X(10) = 4) + P(X(10) = 5)\}$$

$$= \frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} + \frac{e^{-10}(10)^3}{3!} + \frac{e^{-10}(10)^4}{4!} + \frac{e^{-10}(10)^5}{5!}$$

$$= e^{-10} \left[ 1 + 10 + \frac{10^2}{2!} + \frac{10^3}{3!} + \frac{10^4}{4!} + \frac{10^5}{5!} \right]$$

$$= 0.067$$

**3. A fisher man catches fish independently at a Poisson rate of 2 from a large lake with a lot fish. If he starts at 10.00am, what is the hour probability that he catches 1 fish by 10.30am and 3 fishes by noon?**

**Solution:**

Let  $X(t)$  denotes the number of fishes caught by the fisherman in  $(0, t)$ . Then  $\{X(t)\}$  follows a Poisson process. Given  $\lambda = 2$  per hr.

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}; n = 0, 1, 2, \dots, \infty$$

$$= \frac{e^{-2t}(2t)^n}{n!}; n = 0,1,2 \dots \dots \dots \infty \lambda = 2$$

Given a fisherman starts catching at 10.00am and fisherman catches fish independently.

$\therefore P[ \text{he catches one fish in 10.30am and 3 fishes in noon} ]$

$$= P[ \text{he catches one fish in 10.30am} ] P[ \text{he catches 3 fishes in noon} ]$$

$$= P [ \text{he catches one fish in 30mins} ] P [ \text{he catches 3 fishes at 12pm} ]$$

$$= P [ \text{he catches one fish in 30mins} ] P [ \text{he catches 3 fishes in 2 hours} ]$$

$$= P [ \text{he catches one fish by } \frac{1}{2} \text{ an hour} ] P [ \text{he catches 3 fishes in 2 hours} ]$$

[ $\because t$  is in hours]

$$= P \left[ X \left( \frac{1}{2} \right) = 1 \right] P [ X(2) = 3 ] = \frac{e^{-2\frac{1}{2}} \left( 2 \left( \frac{1}{2} \right) \right)^1}{1!} \frac{e^{-4} 4^3}{3!} = e^{-1} \frac{32e^{-4}}{3}$$

$$P [ \text{he catches 3 fishes in noon} ] = 0.07188$$

**4. On the average, a submarine on patrol sights 6 enemy ship per hour.**

**Assume that the number of ships sighted in a given length of time is a**

**Poisson variate, find the probability of sighting (1) 6 ships in the next half**

**an hour (2) 4 ships in the next 2 hours (3) at least one ship in the next 15**

**minutes.**

**Solution:**

Let  $X(t)$  denote the number of submarine on patrol sights in the interval  $(0, t)$ .

Then  $\{X(t)\}$  follows Poisson process.

Given  $\lambda = 6$  per hour

$$P[X(t) = n] = \frac{e^{-\lambda t} (\lambda t)^n}{n!}; n = 0, 1, 2, \dots, \infty$$

$$= \frac{e^{-6t} (6t)^n}{n!}; n = 0, 1, 2, \dots, \infty$$

(i)  $P[\text{sighting 6 ships next half an hour}] = P\left[X\left(\frac{1}{2}\right) = 6\right]$

$$= \frac{e^{-6\left(\frac{1}{2}\right)} \left(6\left(\frac{1}{2}\right)\right)^6}{6!} = \frac{e^{-3} 3^6}{6!} = 0.0504$$

(ii)  $P[\text{sighting 4 ships in the next 2 hours}] = P[X(2) = 4]$

$$= \frac{e^{-6(2)} (6(2))^4}{4!} = \frac{e^{-12} 12^4}{4!} = 0.0053$$

(iii)  $P[\text{sighting at least one ship in the next 15mins}] = P[\text{at least one ship in the next } \frac{1}{4} \text{ hr}]$

$$= P\left[X\left(\frac{1}{4}\right) \geq 1\right] = 1 - P\left[X\left(\frac{1}{4}\right) < 1\right] = 1 - P\left[X\left(\frac{1}{4}\right) = 0\right]$$

$$= 1 - \frac{e^{-6\left(\frac{1}{4}\right)} \left(6\left(\frac{1}{4}\right)\right)^0}{0!}$$

$$= 1 - e^{-\frac{3}{2}} = 0.975$$

Problems under  $P(N(t) = n) = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}$

**1. A radioactive source emits particles at rate of five per minute in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in a 4 minutes period.**

**Solution:**

Given:  $\lambda = 5/\text{min}$  and  $t = 4, n = 10$

Probability of recording :  $p = 0.6$

Let  $N(t)$  be the number of particles recorded during the interval  $(0, t)$

Then  $\{N(t)\}$  follows a Poisson process with parameter  $\lambda p$ .

$$\begin{aligned} P(N(t) = n) &= \frac{e^{-\lambda pt}(\lambda pt)^n}{n!} \\ &= \frac{e^{-3t}(3t)^n}{n!}; n = 0, 1, 2, 3, \dots \end{aligned}$$

$$P(\text{10 particles are recorded in a 4 minute period}) = P(N(4) = 10) = \frac{e^{-12}(12)^{10}}{10!}$$

$$P(N(4) = 10) = 0.1048$$

**2.If customer arrives at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between two**



consecutive arrivals is (i) more than 1 minute (ii) between 1 minute and 2 minutes and (iii) 4 minutes or less.

**Solution:**

Given:  $\lambda = 2/\text{min}$

Let  $T$  be the interval between two consecutive arrivals.

The exponential distribution is  $f(t) = 2e^{-2t}, t > 0$

$$(i) P[T > 1] = \int_1^{\infty} f(t) dt$$

$$P[T > 1] = \int_1^{\infty} 2e^{-2t} dt = 2 \left[ \frac{e^{-2t}}{-2} \right]_1^{\infty} = -[e^{-\infty} - e^{-2}]$$

$$P[T > 1] = e^{-2} = 0.1353$$

$$(ii) P[1 < T < 2] = \int_1^2 f(t) dt$$

$$= \int_1^2 2e^{-2t} dt = 2 \left[ \frac{e^{-2t}}{-2} \right]_1^2 = -[e^{-4} - e^{-2}]$$

$$= -[0.0183 - 0.1353]$$

$$= 0.117$$

$$(iii) P[T \leq 4] = \int_0^4 f(t) dt$$

$$= \int_0^4 2e^{-2t} dt = 2 \left[ \frac{e^{-2t}}{-2} \right]_0^4 = -[e^{-8} - e^{-0}]$$

$$= -[3.35 * 10^{-4} - 1]$$

$$= 0.996$$

$$P[T \leq 4] = 0.996$$

