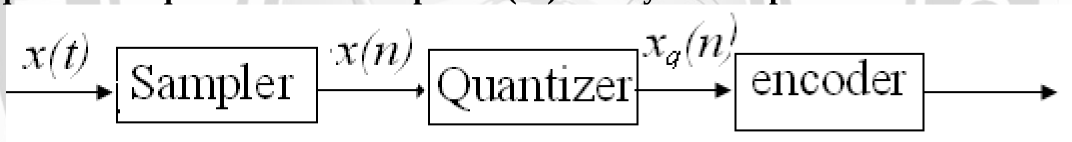
QUANTIZATION NOISE,COEFFICIENT QUANTIZATION ERROR

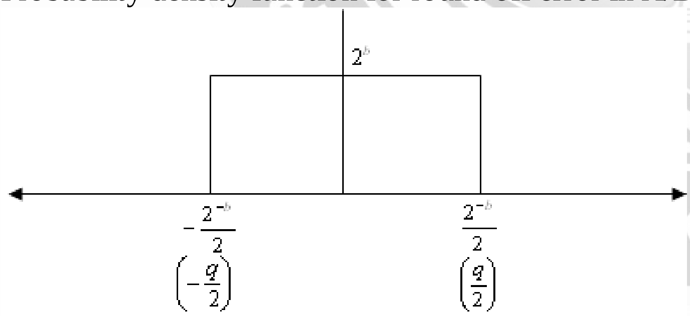
Quantization Noise power:

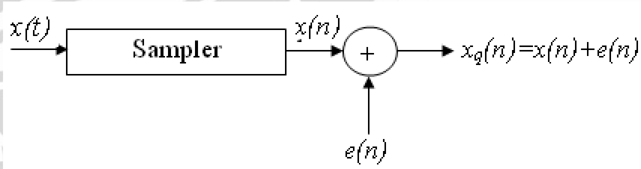
<u>Input Quantization error:</u>

***Derive the equation for quantization noise power (or) Steady state Input Noise Power.**



Probability density function for round off error in A/D conversion is





If rounding is used for quantization, which is bounded by $-\frac{q}{2} \le e(n) \le \frac{q}{2}$, then the error lies between $-\frac{q}{2}$ to $\frac{q}{2}$ with equal probability, where q→ quantization step size.

<u>Properties of analog to digital conversion error, e(n):</u>

1. The error sequence e(n) is a sample sequence of a stationary random process.
2. The error sequence is uncorrelated with x(n) and other signals in the system.
3. The error is a white noise process with uniform amplitude probability distribution over the range of quantization error.

The variance of e(n) is given by

$$\sigma_e^2 = E\left[e^2(n)\right] - E^2\left[e(n)\right] \text{ ---------------------------->(1)}$$

Where $E\left[e^2(n)\right]$→Average of $e^2(n)$

$E\left[e(n)\right]$→Mean value of e(n).

For rounding, e(n) lies between $-\frac{q}{2}$ and $\frac{q}{2}$ with equal probability

$$E\left[e^2(n)\right] = \int_{-\infty}^{\infty} e^2(n)p(e)de \text{ ------------------------------>(2)}$$

$$p(e) = \frac{1}{q},\ -\frac{q}{2} \le e(n) \le \frac{q}{2} \text{ ------------------------------>(3)}$$

Substituting (3) in (2)

$$E\left[e^2(n)\right] = \int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n)\frac{1}{q}de$$

$$E\left[e^2(n)\right] = \frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n)de \text{ ------------------------------------>(4)}$$

$$E\left[e(n)\right] = 0$$

$$E^2\left[e(n)\right] = 0 \text{ --->(5)}$$

Substituting (4) and (5) in (1)

$$\sigma_e^2 = \frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n)de - 0$$

$$= \frac{1}{q}\left[\frac{e^3}{3}\right]_{-\frac{q}{2}}^{\frac{q}{2}}$$

$$= \frac{1}{3q}\left[\left(\frac{q}{2}\right)^3 - \left(-\frac{q}{2}\right)^3\right]$$

$$= \frac{1}{3q}\left[\left(\frac{q^3}{8}\right) - \left(-\frac{q^3}{8}\right)\right]$$

$$= \frac{1}{3q}\left[\left(\frac{q^3}{8}\right) + \left(\frac{q^3}{8}\right)\right]$$

$$= \frac{1}{3q}\left[\frac{2q^3}{8}\right]$$

$$\sigma_e^2 = \frac{q^2}{12} \text{ -->(6)}$$

In general,

$$\frac{1}{2^b} = 2^{-b} = q \text{ -->(7)}$$

$$\sigma_e^2 = \frac{\left(2^{-b}\right)^2}{12}$$

$$\sigma_e^2 = \frac{2^{-2b}}{12} \text{ -->(8)}$$

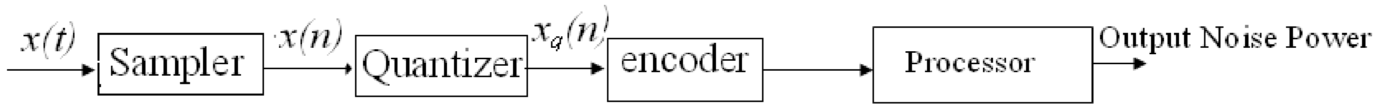
Equation (8) is known as the steady state noise power due to input quantization.

$q = \frac{R}{2^b}$ → in two's complement representation.

$q = \frac{R}{2^b - 1}$ → in sign magnitude (or) one's complement representation.

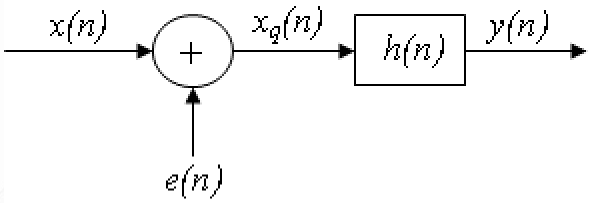
R → Range of analog signal to be quantized.

Steady state Output Noise power:



After quantization, we have noise power σ_e^2 as input noise power. Therefore, Output noise power of system is given by

$$\sigma_{eo}^2 = \sigma_e^2\left[\sum_{n=0}^{\infty} h^2(n)\right] \text{------------------------------------>(9)}$$



where h(n) → impulse response of the system.

Let error E(n) be output noise power due to quantization

Error $E(n) = e(n) * h(n)$

$$= \sum_{k=0}^{\infty} h(n)e(n-k)$$

The variance of error E(n) is called output noise power, σ_e^2.

By using Parseval's theorem,

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n)$$

$$= \sigma_e^2 \frac{1}{2\pi j}\oint H(Z)H(Z^{-1})\frac{dZ}{z}$$

Where the closed contour integration is evaluated using the method of residue by taking only the poles that lie inside the unit circle.

Z transform of h(n), $H(Z) = \sum_{n=0}^{\infty} h(n)z^{-n}$

Z transform of $h^2(n)$ = Z[$h^2(n)$] $= \sum_{n=0}^{\infty} h^2(n)z^{-n} = \sum_{n=0}^{\infty} h(n)h(n)z^{-n}$ --------------------->(10)

By Inverse Z transform, $h(n) = \frac{1}{2\pi j}\oint H(Z)Z^{n-1}dZ$ --------------------------------->(11)

Substituting (11) in (10)

$$\sum_{n=0}^{\infty} h^2(n)z^{-n} = \sum_{n=0}^{\infty}\frac{1}{2\pi j}\oint H(Z)Z^{n-1}dZ \;\; h(n)z^{-n}$$

$$= \frac{1}{2\pi j}\oint H(Z)\left[\sum_{n=0}^{\infty} h(n)Z^{-1}\right]dZ$$

$$\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j}\oint H(Z)\left[\sum_{n=0}^{\infty} h(n)Z^{-1}\right]\frac{dZ}{Z^{-n}}$$

$$= \frac{1}{2\pi j}\oint H(Z)\left[\sum_{n=0}^{\infty} h(n)(Z^{-n})^{-1}Z^{-1}dZ\right]$$

$$\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j}\oint H(Z)H(Z^{-1})\frac{dZ}{Z} \text{----------------------------------->(12)}$$

Substituting (12) in (9)

$$\sigma_{eo}^2 = \sigma_e^2\left[\frac{1}{2\pi j}\oint H(Z)H(Z^{-1})Z^{-1}dZ\right]$$

Problem:

The output signal of an A/D converter is passed through a first order low pass filter, with transfer function given by

$H(z)=\frac{(1-a)z}{z-a}$ *for* $0<a<1$. **Find the steady state output noise power due to quantization at the output of the digital filter. [Nov/Dec-2015]**

Solution:

$$\sigma_e{}^2=\sigma_e{}^2\frac{1}{2\pi j}\oint_c H(z)H(z^{-1})z^{-1}dz$$

$$Given \quad H(z)=\frac{(1-a)z}{(z-a)} \qquad H(z^{-1})=\frac{(1-a)z^{-1}}{(z^{-1}-a)}$$

Substituting $H(z)$ *and* $H(z^{-1})$ *in equation (1), we have*

$$\sigma_e{}^2=\frac{\sigma_e{}^2}{2\pi j}\oint_c\frac{(1-a)z}{(z-a)}\frac{(1-a)z^{-1}}{(z^{-1}-a)}z^{-1}dz=\frac{\sigma_e{}^2}{2\pi j}\oint_c\frac{(1-a)^2}{(z-a)}\frac{dz}{(z^{-1}-a)}z^{-1}$$

$$=\sigma_e{}^2\left[residue\ of\ H(z)\,H(z^{-1})z^{-1}at\ z=a\ +residue\ of\ H(z)\,H(z^{-1})z^{-1}at\ z=\frac{1}{a}\right]$$

$$=\sigma_e{}^2\left[(z-a)\frac{(1-a)^2z^{-1}}{(z-a)(z^{-1}-a)}+0\right]$$

$$=\sigma_e{}^2\left[\frac{(1-a)^2}{(z^{-1}-a)}\right]=\sigma_e{}^2\left[\frac{(1-a)}{(1+a)}\right]$$

Where, $\sigma_e{}^2=\frac{2^{-2b}}{12}$

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Find the steady state variance of the noise in the output due to quantization of input for the first order filter. [Apr/May'11] [Nov/Dec-2016]

$$y(n)=ay(n-1)+x(n)$$

Solution:

The impulse response for the above filter is given by $h(n)=a^nu(n)$

$$\sigma_\epsilon^2=\sigma_e^2\sum_{k=0}^{\alpha}h^2(n)$$

$$=\sigma_e^2\sum_{k=0}^{\alpha}a^{2n}$$

$$=\sigma_e^2\left[1+a^2+a^4+....\infty\right]$$

$$=\sigma_e^2\frac{1}{1-a^2}$$

$$=\frac{2^{-2b}}{12}\left[\frac{1}{1-a^2}\right] \qquad (or)$$

Taking Z-transform on both sides we have

$$Y(z)=az^{-1}Y(z)+X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1-az^{-1}}=\frac{z}{z-a}$$

$$H(z^{-1})=\frac{z^{-1}}{z^{-1}-a}$$

We know

$$\sigma_{\epsilon}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z) H(z^{-1}) z^{-1} dz$$

Substituting $H(z)$ and $H(z^{-1})$ values in the above equation we get

$$\sigma_{\epsilon}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c \frac{z}{z-a} \frac{z^{-1}}{z^{-1}-a} z^{-1} dz$$

$$\sigma_{\epsilon}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c \frac{z^{-1}}{(z-a)(z^{-1}-a)} dz$$

$$= \sigma_e^2 \left[\begin{array}{l} \text{residue of } \dfrac{z^{-1}}{(z-a)(z^{-1}-a)} \text{ at } z=a \\ +\text{residue of } \dfrac{z^{-1}}{(z-a)(z^{-1}-a)} \text{ at } z=1/a \end{array} \right]$$

$$= \sigma_e^2 \left[(z-a) \frac{z^{-1}}{(z-a)(z^{-1}-a)} \Bigg|_{z=a} \right]$$

$$= \sigma_e^2 \frac{a^{-1}}{a^{-1}-a} = \sigma_e^2 \frac{1}{1-a^2}$$

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The output of the A/D converter is applied to a digital filter with the system function

$$H(Z) = \frac{0.45Z}{Z-0.72}$$

Find the output noise power of the digital filter, when the input signal is quantized to 7 bits.

Given:

$$H(Z) = \frac{0.45Z}{Z-0.72}$$

Solution:

$$H(Z)H(Z^{-1})Z^{-1} = \frac{0.45Z}{Z-0.72} \times \frac{0.45Z^{-1}}{Z^{-1}-0.72} \times Z^{-1}$$

$$= \frac{0.45^2 Z^{-1}}{(Z-0.72)\left(\frac{1}{Z}-0.72\right)}$$

$$= \frac{0.2025Z^{-1}}{(Z-0.72)\left(\frac{1-0.72Z}{Z}\right)}$$

$$= \frac{0.2025Z^{-1}Z}{(Z-0.72)\left(Z-\frac{1}{0.72}\right)}$$

$$= \frac{-0.28125}{(Z-0.72)(Z-1.3889)}$$

Now the poles of $H(Z)H(Z^{-1})Z^{-1}$ are $p_1=0.72$, $p_2=1.3889$

Output noise power due to input quantization

$$\sigma_{eo}^2 = \sigma_e^2 \left[\frac{1}{2\pi j} \oint H(Z) H(Z^{-1}) Z^{-1} dZ \right]$$

$$= \sigma_e^2 \sum_{i=1}^{N} \operatorname{Res}\left[H(Z) H(Z^{-1}) Z^{-1} \right]\Big|_{z=p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \operatorname{Res}\left[H(Z)H(Z^{-1})Z^{-1}\right]\Big|_{z=p_i}$$

Where $p_1, p_2, \ldots p_n$ are the poles of $H(Z)H(Z^{-1})\, Z^{-1}$ that lies inside the unit circle in z-plane.

$$\sigma_{eo}^2 = \sigma_e^2 \times (Z-0.72) \times \frac{-0.28125}{(Z-0.72)(Z-1.3889)}\Bigg|_{Z=0.72}$$

$$= \sigma_e^2 \times \frac{-0.28125}{0.72-1.3889}$$

$$= 0.4205\sigma_e^2$$

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Consider the transfer function $H(z) = H_1(z)H_2(z)$ where $H_1(z) = \frac{1}{1-a_1z^{-1}}$ and $H_2(z) = \frac{1}{1-a_2z^{-1}}$

Find the output round off noise power. Assume $\alpha_1 = 0.5$ and $\alpha_2 = 0.6$ and find output round off noise power.

Solution:

The round off noise model for $H(z) = H_1(z)H_2(z)$ is given by,

From the realization we can find that the noise transfer function seen by noise source $e_1(n)$ is H(z), where,

$$H(z) = \frac{1}{(1-a_1z^{-1})(1-a_2z^{-1})} ------------------------(1)$$

Whereas, the noise transfer function seen by $e_2(n)$ is,

$$H_2(z) = \frac{1}{(1-a_2z^{-1})} ------------------------------(2)$$

The total steady state noise variance can be obtained, we have

$$\sigma_0^{\,2} = \sigma_{01}^{\,2} + \sigma_{02}^{\,2} ------------------------------(3)$$

$$\sigma_{01}^{\,2} = \frac{1}{2\pi j}\oint_c H(z)H(z^{-1})z^{-1}dz$$

$$= \sigma_e^{\,2}\frac{1}{2\pi j}\oint_c \frac{1}{1-a_1z^{-1}}\frac{1}{1-a_2z^{-1}}\frac{1}{1-a_1z}\frac{1}{1-a_2z}z^{-1}dz$$

$$= \sigma_e^{\,2}\left[\sum \textit{of residue of } H(z)H(z^{-1})z^{-1} \textit{ at poles } z=a_1, z=a_2, z=\frac{1}{a_1} \textit{ and } z=\frac{1}{a_2}\right]$$

If a1 and a2 are less than the poles $z=1/a_1$ and $z=1/a_2$ lies outside of the circle $|z|=1$. So, the residue of H(z) $H(z^{-1})\, z^{-1}$ at $z=1/a_1$ and $z=1/a_2$ are zero. Consequently we have,

$$\sigma_{01}^{\,2} = \left[\sum \textit{of residue of } H(z)H(z^{-1})z^{-1} \textit{ at poles } z=a_1, z=a_2\right]$$

$$= \left[(z-a_1)\frac{z^{-1}}{(1-a_1z^{-1})(1-a_2z^{-1})(1-a_1z)(1-a_2z)}\Bigg|_{z=a_1} + (z-a_2)\frac{z^{-1}}{(1-a_1z^{-1})(1-a_2z^{-1})(1-a_1z)(1-a_2z)}\Bigg|_{z=a_2}\right]$$

$$= \sigma_e^{\,2}\left[\frac{1}{\left(1-\frac{a_2}{a_1}\right)(1-a_2^{\,2})(1-a_1a_2)} + \frac{1}{\left(1-\frac{a_2}{a_1}\right)(1-a_1a_2)(1-a_2^{\,2})}\right]$$

$$\sigma_{01}^{\,2} = \sigma_e^{\,2}\left[\frac{a_1}{a_1-a_2}\cdot\frac{1}{1-a_1^{\,2}}\cdot\frac{1}{1-a_1a_2} + \frac{a_2}{a_2-a_1}\cdot\frac{1}{1-a_2^{\,2}}\cdot\frac{1}{1-a_1a_2}\right] --------------(4)$$

In the same way,

$$\sigma_{02}^{\ 2} = \frac{\sigma_e^{\ 2}}{2\pi j}\oint_c H_2(z)H_2(z^{-1})z^{-1}dz$$

$$= \frac{\sigma_e^{\ 2}}{2\pi j}\oint_c \frac{1}{1-a_2z^{-1}}\frac{1}{1-a_2z}z^{-1}dz$$

$$= \sigma_e^{\ 2}\left[(z-a_2)\frac{z^{-1}}{(1-a_2z^{-1})(1-a_2z)}\bigg|_{z=a_2}\right]$$

$$= \sigma_e^{\ 2}\left[(z-a_2z^{-1})\frac{z^{-1}}{(1-a_2z^{-1})(1-a_2z)}\bigg|_{z=a_2}\right]$$

$$= \sigma_e^{\ 2}\left[\frac{1}{1-a_2^{\ 2}}\right] \text{------------------------------------(5)}$$

$$\sigma_0^{\ 2} = \sigma_e^{\ 2}\left[\frac{1}{1-a_2^{\ 2}} + \frac{a_1}{a_1-a_2}.\frac{1}{1-a_1^{\ 2}}.\frac{1}{1-a_1a_2} + \frac{a_2}{a_2-a_1}.\frac{1}{1-a_2^{\ 2}}.\frac{1}{1-a_1a_2}\right]$$

$$= \sigma_e^{\ 2}\left[\frac{1}{1-a_2^{\ 2}} + \frac{a_1(1-a_2^{\ 2})-a_2^{\ 2}(1-a_1^{\ 2})}{(1-a_1^{\ 2})(1-a_2^{\ 2})(1-a_1a_2)(a_1-a_2)}\right]$$

$$= \sigma_e^{\ 2}\left[\frac{1}{1-a_2^{\ 2}} + \frac{(a_1-a_2)(1+a_1a_2)}{(1-a_1^{\ 2})(1-a_2^{\ 2})(1-a_1a_2)(a_1-a_2)}\right]$$

$$= \frac{2^{-2b}}{12}\left[\frac{1}{1-a_2^{\ 2}} + \frac{(1+a_1a_2)}{(1-a_1^{\ 2})(1-a_2^{\ 2})(1-a_1a_2)}\right]$$

The steady state noise power for $a_1 = 0.5, a_2 = 0.6$ is given by

$$= \frac{2^{-2b}}{12}\left[\frac{1}{1-(0.6)^2} + \frac{1+(0.5)(0.6)}{(1-(0.5)^2)(1-(0.6)^2)(1-0.6(0.5))}\right]$$

$$= \frac{2^{-2b}}{12}(5.4315)$$

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Draw the quantization noise model for a second order system $H(z) = \frac{1}{1-2r\cos\theta z^{-1} + r^2z^{-2}}$ **and find the steady state output noise variance.**

Solution:

Given:

$$H(z) = \frac{1}{1-2r\cos\theta z^{-1} + r^2z^{-2}}$$

The quantization noise model is,

we know, $\sigma_0^{\ 2} = \sigma_{01}^{\ 2} + \sigma_{02}^{\ 2}$

Both noise sources see the same transfer function

$$H(z) = \frac{1}{1-2r\cos\theta z^{-1} + r^2z^{-2}}$$

The impulse response of the transfer function is given by

$$h(n) = r^n\frac{\sin(n+1)\theta}{\sin\theta}u(n)$$

Now the steady state output noise variance is,

$$\sigma_0^{\ 2} = \sigma_{01}^{\ 2} + \sigma_{02}^{\ 2}$$

But $\sigma_{01}{}^2 = \sigma_{02}{}^2 = \sigma_e{}^2 \sum_{n=-\infty}^{\infty} h^2(n)$, *which gives us*

$$\sigma_0{}^2 = 2.\frac{2^{-2b}}{12}\sum_{n=0}^{\infty} r^{2n}\,\frac{\sin^2(n+1)\theta}{\sin^2\theta}$$

$$= 2.\frac{2^{-2b}}{12}\frac{1}{2\sin^2\theta}\sum_{n=0}^{\infty} r^{2n}\left[1-\cos 2(n+1)\theta\right] \qquad \therefore \cos 2\theta = 1-2\sin^2\theta$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\sum_{n=0}^{\infty} r^{2n} - \sum_{n=0}^{\infty} r^{2n}\cos 2(n+1)\theta\right]$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\frac{1}{1-r^2} - \frac{1}{2}\left(\sum_{n=0}^{\infty} r^{2n}e^{j2(n+1)\theta} + \sum_{n=0}^{\infty} r^{2n}e^{-j2(n+1)\theta}\right)\right]$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\frac{1}{1-r^2} - \frac{1}{2}\left(\frac{e^{j2\theta}}{1-r^2e^{2j\theta}} + \frac{e^{-j2\theta}}{1-r^2e^{-2j\theta}}\right)\right]$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\frac{1}{1-r^2} - \frac{\cos 2\theta - r^2}{1-2r^2\cos 2\theta + r^4}\right]$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\frac{(1+r)^2(1-\cos 2\theta)}{(1-r^2)(1-2r^2\cos 2\theta + r^4)}\right]$$

$$= \frac{2^{-2b}}{6}\frac{(1+r)^2}{(1-r^2)(1-2r^2\cos 2\theta + r^4)}$$

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