Co-ordinate Systems

In order to describe the spatial variations of the quantities, we require using appropriate co-ordinate system. A point or vector can be represented in a **curvilinear** coordinate system that may be **orthogonal** or **non-orthogonal**.

An orthogonal system is one in which the co-ordinates are mutually perpendicular. Non-orthogonal co-ordinate systems are also possible, but their usage is very limited in practice .

Let *u* = constant, *v* = constant and *w* = constant represent surfaces in a coordinate system, the surfaces may be curved surfaces in general. Furthur, let

t $\hat{a}_u$, $\hat{a}_v$ and $\hat{a}_w$

be the unit vectors in the three coordinate directions(base vectors). In a general right handed orthogonal curvilinear systems, the vectors satisfy the following relations :

$$\hat{a}_u \times \hat{a}_v = \hat{a}_w$$
$$\hat{a}_v \times \hat{a}_w = \hat{a}_u$$
$$\hat{a}_w \times \hat{a}_u = \hat{a}_v \quad \text{.......................................(1.13)}$$

These equations are not independent and specification of one will automatically imply the other two. Furthermore, the following relations hold

$$\hat{a}_u \cdot \hat{a}_v = \hat{a}_v \cdot \hat{a}_w = \hat{a}_w \cdot \hat{a}_u = 0$$
$$\hat{a}_u \cdot \hat{a}_u = \hat{a}_v \cdot \hat{a}_v = \hat{a}_w \cdot \hat{a}_w = 1 \quad \text{.................(1.14)}$$

A vector can be represented as sum of its orthogonal components, $\vec{A} = A_u\,\hat{a}_u + A_v\,\hat{a}_v + A_w\,\hat{a}_w$ ....(1.15)

In general *u*, *v* and *w* may not represent length. We multiply *u*, *v* and *w* by conversion factors *h*1,*h*2 and *h*3 respectively to convert differential changes d*u*, d*v* and d*w* to corresponding changes in length d*l*1, d*l*2, and d*l*3. Therefore

$$d\vec{l} = \hat{a}_u\, dl_1 + \hat{a}_v\, dl_2 + \hat{a}_w\, dl_3$$
$$= h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w \quad \text{...............(1.16)}$$

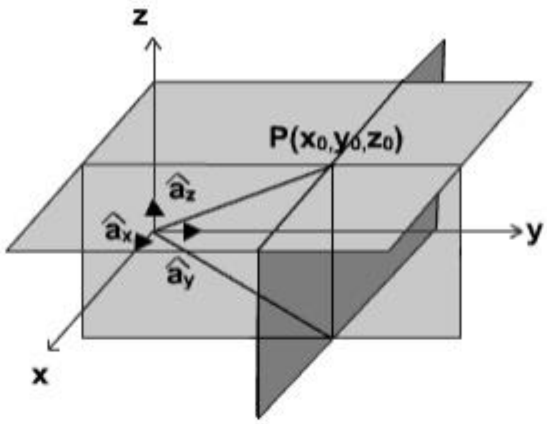
In the same manner, differential volume dv can be written as $dv = h_1 h_2 h_3\, du\, dv\, dw$ and differential area ds_1 normal to $\hat{a}_u$ is given by, $ds_1 = h_2 h_3\, dv\, dw$. In the same manner, differential areas normal to unit vectors $\hat{a}_v$ and $\hat{a}_w$ can be defined.

In the following sections we discuss three most commonly used orthogonal co-ordinate systems, viz:

1. **Cartesian (or rectangular) co-ordinate system**
2. **Cylindrical co-ordinate system**
3. **Spherical polar co-ordinate system**

Cartesian Co-ordinate System :

In Cartesian co-ordinate system, we have, $(u,v,w) = (x,y,z)$. A point $P(x0, y0, z0)$ in Cartesian co-ordinate system is represented as intersection of three planes $x = x0$, $y = y0$ and $z = z0$. The unit vectors satisfies the following relation:



Cartesian Coordinate System

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$

$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$

$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$$

$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$$

$$\overrightarrow{OP} = \hat{a}_x x_0 + \hat{a}_y y_0 + \hat{a}_z z_0$$

In cartesian co-ordinate system, a vector $\vec{A}$ can be written as $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$. The dot and cross product of two vectors $\vec{A}$ and $\vec{B}$ can be written as follows:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \qquad \text{.................(1.19)}$$

$$\vec{A} \times \vec{B} = \hat{a}_x (A_y B_z - A_z B_y) + \hat{a}_y (A_z B_x - A_x B_z) + \hat{a}_z (A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \text{.....................(1.20)}$$

Since *x*, *y* and *z* all represent lengths, $h1 = h2 = h3 = 1$. The differential length, area and volume are defined respectively as

$$d\vec{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z \quad ....(1.21)$$

$$d\vec{s}_x = dydz\,\hat{a}_x$$

$$d\vec{s}_y = dxdz\,\hat{a}_y$$

$$d\vec{s}_z = dxdy\,\hat{a}_z$$

$$d\upsilon = dxdydz \quad ....(1.22)$$

Cylindrical Co-ordinate System :

For cylindrical coordinate systems we have $(u,v,w) = (r,\phi,z)$ a point $P(r_0,\phi_0,z_0)$ is determined as the point of intersection of a cylindrical surface $r = r_0$, half plane containing the z-axis and making an angle $\phi = \phi_0$; with the xz plane and a plane parallel to *xy* plane located at $z=z_0$ as shown in figure 7 on next page.

In cylindrical coordinate system, the unit vectors satisfy the following relations

A vector $\vec{A}$ can be written as , $\vec{A} = A_\rho\,\hat{a}_\rho + A_\phi\,\hat{a}_\phi + A_z\,\hat{a}_z$ ....(1.24)

The differential length is defined as,

$$d\vec{l} = \hat{a}_\rho\,d\rho + \rho d\phi\,\hat{a}_\phi + dz\,\hat{a}_z \qquad h_1 = 1, h_2 = \rho, h_3 = 1 \quad ....(1.25)$$

$$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z$$

$$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho$$

$$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi$$

.....................(1.23)

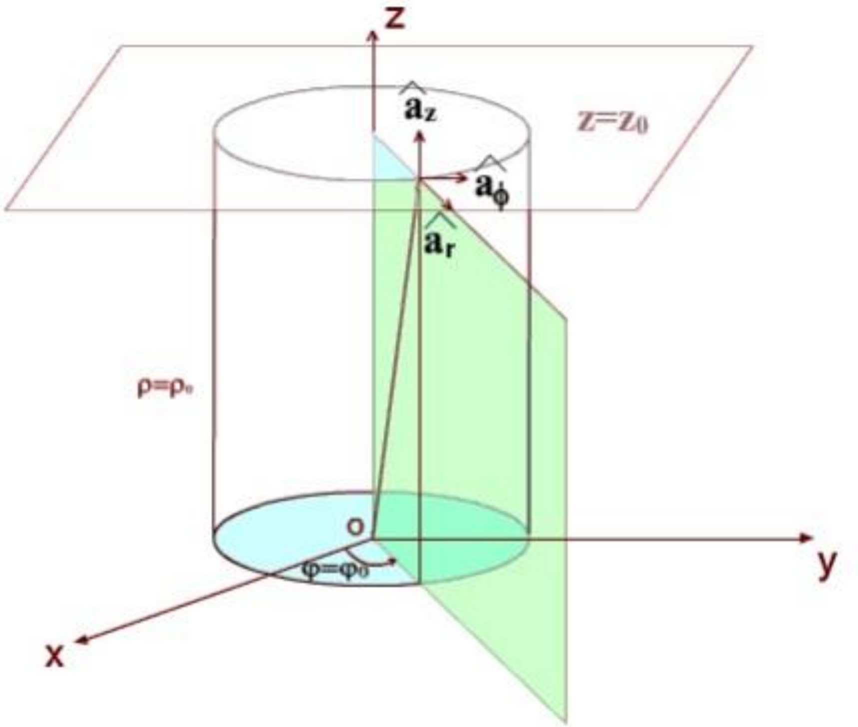


Fig 1.7 : Cylindrical Coordinate System

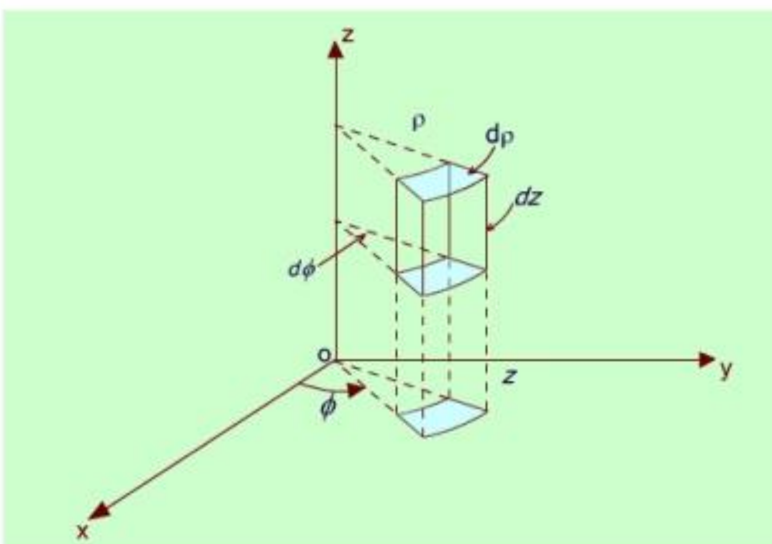


Fig 1.8 : Differential Volume Element in Cylindrical Coordinates

Differential areas are:

$$\overrightarrow{ds_\rho} = \rho d\phi dz \, \hat{a}_\rho$$

$$\overrightarrow{ds_\phi} = d\rho dz \, \hat{a}_\phi \quad \text{..............(1.26)}$$

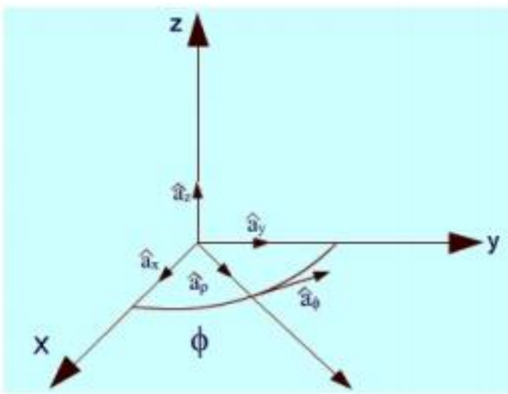
$$\overrightarrow{ds_z} = \rho d\phi d\rho \, \hat{a}_z$$

Differential volume,

$$d\upsilon = \rho d\rho d\phi dz \quad \text{................(1.27)}$$

Transformation between Cartesian and Cylindrical coordinates:

Let us consider $\vec{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z$ is to be expressed in Cartesian co-ordinate as $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$. In doing so we note that $A_x = \vec{A}.\hat{a}_x = \left(\hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z\right).\hat{a}_x$ and it applies for other components as well.



$$\hat{a}_\rho.\hat{a}_x = \cos\phi$$
$$\hat{a}_\rho.\hat{a}_y = \sin\phi \quad \text{..............(1.28)}$$
$$\hat{a}_\phi.\hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi$$
$$\hat{a}_\phi.\hat{a}_y = \cos\phi$$

Therefore we can write,

$$A_x = \vec{A}.\hat{a}_x = A_\rho\cos\phi - A_\phi\sin\phi$$
$$A_y = \vec{A}.\hat{a}_y = A_\rho\sin\phi + A_\phi\cos\phi \quad \text{.........(1.29)}$$
$$A_z = \vec{A}.\hat{a}_z = A_z$$

Fig 1.9 : Unit Vectors in Cartesian and Cylindrical Coordinates

These relations can be put conveniently in the matrix form as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_y \\ A_z \end{bmatrix} \text{....................(1.30)}$$

A_ρ, A_ϕ and A_z themselves may be functions of ρ, ϕ and z as:

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z \quad \text{............................(1.31)}$$

The inverse relationships are:

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z \quad \text{........................(1.32)}$$

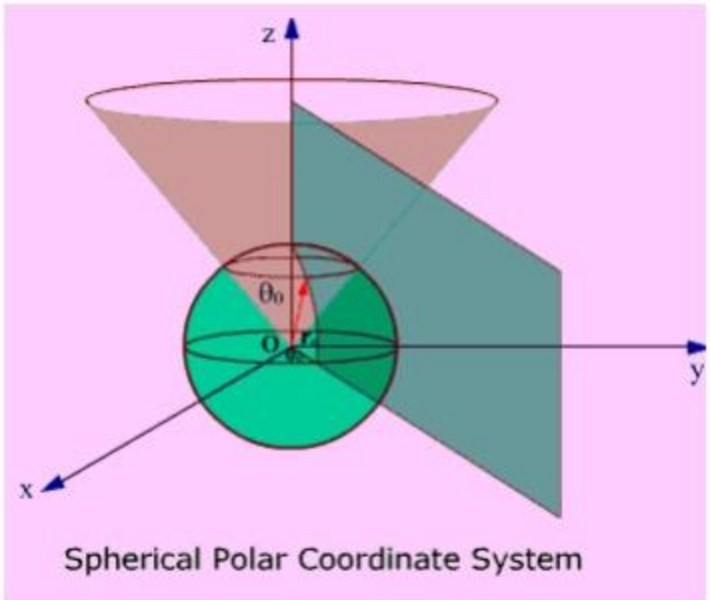


Fig 1.10: Spherical Polar Coordinate System

Thus we see that a vector in one coordinate system is transformed to another coordinate system through two-step process: Finding the component vectors and then variable transformation.

Spherical Polar Coordinates:

For spherical polar coordinate system, we have, $(u, v, w) = (r, \theta, \phi)$. A point $P(r_0, \theta_0, \phi_0)$ is represented as the intersection of

(i) Spherical surface $r=r_0$

(ii) Conical surface $\theta = \theta_0$,and

(iii) half plane containing z-axis making angle $\phi = \phi_0$ with the *xz* plane as shown in the figure 1.10.

The unit vectors satisfy the following relationships:

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$$
$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$$
$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta \quad \text{.....................................(1.33)}$$

The orientation of the unit vectors are shown in the figure 1.11.

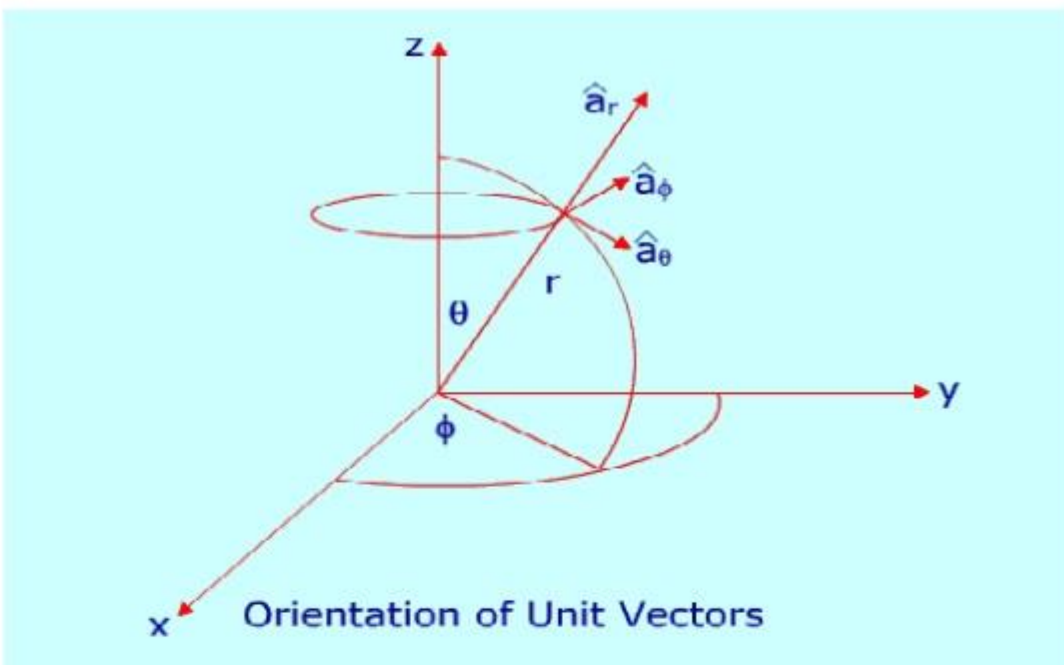


Fig 1.11: Orientation of Unit Vectors

A vector in spherical polar co-ordinates is written as : $\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$ and $d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r \sin\theta d\phi$

For spherical polar coordinate system we have h_1=1, h_2= r and h_3=$r\sin\theta$.

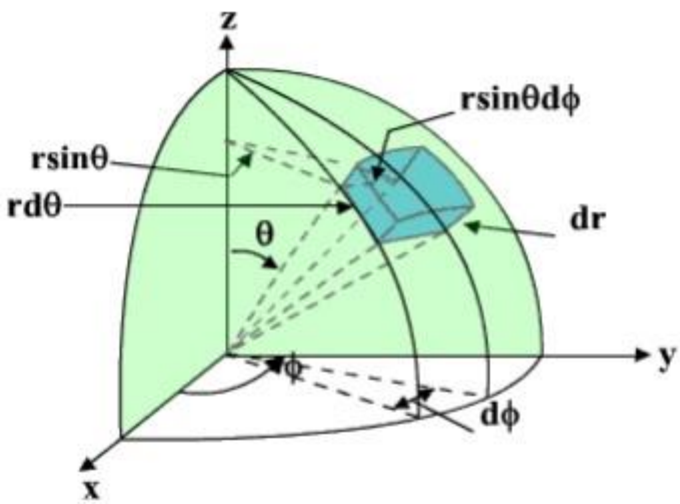


Fig 1.12(a) : Differential volume in s-p coordinates

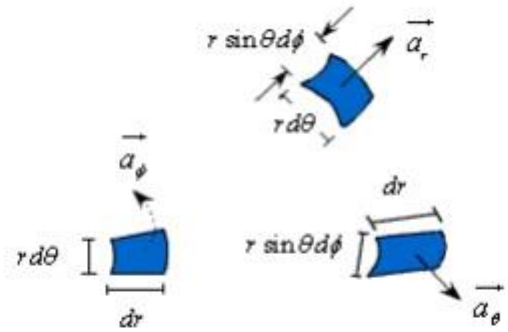


Fig 1.12(b) : Exploded view

With reference to the Figure 1.12, the elemental areas are:

$$ds_r = r^2 \sin\theta d\theta d\phi \hat{a}_r$$
$$ds_\theta = r \sin\theta dr d\phi \hat{a}_\theta$$
$$ds_\rho = r dr d\theta \hat{a}_\phi \quad \text{........................(1.34)}$$

and elementary volume is given by

$$d\upsilon = r^2 \sin\theta dr d\theta d\phi \text{ ....(1.35)}$$

Coordinate transformation between rectangular and spherical polar:

With reference to the Figure 1.12, the elemental areas are:

$$\hat{a}_r \cdot \hat{a}_x = \sin\theta\cos\phi$$

$$\hat{a}_r \cdot \hat{a}_y = \sin\theta\sin\phi$$

$$\hat{a}_r \cdot \hat{a}_z = \cos\theta$$

$$\hat{a}_\theta \cdot \hat{a}_x = \cos\theta\cos\phi$$

$$\hat{a}_\theta \cdot \hat{a}_y = \cos\theta\sin\phi$$

$$\hat{a}_\theta \cdot \hat{a}_z = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$$

$$\hat{a}_\phi \cdot \hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi$$

$$\hat{a}_\phi \cdot \hat{a}_y = \cos\phi$$

$$\hat{a}_\phi \cdot \hat{a}_z = 0$$

..(1.36)

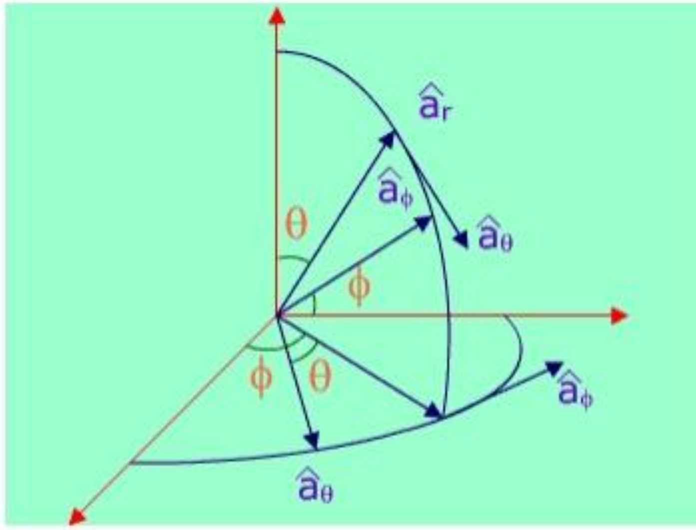


Fig 1.13: Coordinate transformation

Given a vector $\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$ in the spherical polar coordinate system, its component in the cartesian coordinate system can be found out as follows:

$$A_x = \vec{A}.\hat{a}_x = A_r \sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi \quad \text{.................................(1.37)}$$

Similarly,

$$A_y = \vec{A}\,\hat{a}_y = A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi \quad \text{.................................(1.38a)}$$

$$A_z = \vec{A}\,\hat{a}_z = A_r \cos\theta - A_\theta \sin\theta \quad \text{.................................(1.38b)}$$

The above equation can be put in a compact form:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \quad \text{.................................(1.39)}$$

The components A_r, A_θ and A_ϕ themselves will be functions of r, θ and ϕ. r, θ and ϕ are related to *x*,*y* and *z* as:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta \quad \text{....................(1.40)}$$

and conversely,

$$r = \sqrt{x^2 + y^2 + z^2} \quad \text{.......................................(1.41a)}$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad \text{.................................(1.41b)}$$

$$\phi = \tan^{-1}\frac{y}{x} \quad \text{...(1.41c)}$$

Using the variable transformation listed above, the vector components, which are functions of variables of one coordinate system, can be transformed to functions of variables of other coordinate system and a total transformation can be done.