3.4 EIGENVALUES OF A MATRIX BY POWER METHOD

1. Find the numerically largest Eigenvalue of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by power method.

Solution

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial Eigenvector.

$$AX_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0.2 \\ 0.7 \\ 1 \end{bmatrix} = 6X_2$$

$$AX_2 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 2.6 \\ 8.3 \end{bmatrix} = 8.3 \begin{bmatrix} 0 \\ 0.3 \\ 1 \end{bmatrix} = 8.3X_3$$

$$AX_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0.3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0.2 \\ 5.9 \end{bmatrix} = 5.9 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} = 5.9X_4$$

$$AX_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.2 \\ -0.2 \\ 6.2 \end{bmatrix} = 6.2 \begin{bmatrix} 0.4 \\ 0 \\ 1 \end{bmatrix} = 6.2X_5$$

$$AX_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 0.6 \\ 7.4 \end{bmatrix} = 7.4 \begin{bmatrix} 0.3 \\ 0.1 \\ 1 \end{bmatrix} = 7.4X_6$$

$$AX_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.6 \\ 7.1 \end{bmatrix} = 7.1 \begin{bmatrix} 0.3 \\ 0.1 \\ 1 \end{bmatrix} = 7.1X_7$$

$X_6 = X_7$, Hence, the numerically largest Eigenvalue =7 and the corresponding Eigenvector $= \begin{bmatrix} 0.3 \\ 0.1 \\ 1 \end{bmatrix}$

2. Find the dominant Eigenvalue and the corresponding Eigenvector of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find also the least latent root and hence the third Eigenvalue also.

Solution

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial Eigenvector.

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\ X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = 7\ X_3$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 1.8 \\ 0 \end{bmatrix} = 3.4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 3.4\, X_4$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4\, X_5$$

$X_4 = X_5$, Hence, the numerically largest Eigenvalue=7 and the corresponding Eigenvector $= \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

$\therefore$ Dominant Eigenvalue= 4 ; corresponding Eigenvector is $(1 \quad 0.5 \quad 0)$

To find the least Eigenvalue, let $B = A - 4I$ since $\lambda_1 = 4$

$$\therefore B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

We will find the dominant Eigenvalue of B

Let $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial Eigenvector.

$$BY_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -3\, Y_2$$

$$BY_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5\, Y_3$$

$$BY_3 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5\, Y_4$$

$$Dominant\ Eigenvalue\ of\ B\ is - 5.$$

Adding 4, smallest Eigenvalue of $A = -5 + 4 = -1$

Sum of the Eigenvalues=Trace of $A = 1 + 2 + 3 = 6$

$$4 + (-1) + \lambda_3 = 6,$$

$\therefore \lambda_3 = 3.$

All the three Eigenvalues are $4, 3, -1$