Gram Schmidt Orthogonalisation Process

Theorem:

Every finite dimensional inner product space has an orthonormal basis.(Gram Schmidt Orthogonalization Process)

PROBLEMS

Apply Gram-Schmidt process to construct an orthonormal basis for $V_3(R)$ with the standard inner product for the basis (v_1,v_2,v_3), Where $v_1 = (1,0,1)$; $v_2 = (1,3,1)$ and $v_3 = (3,2,1)$.

Sol: The first vector in the orthogonal basis is

$$w_1 = v_1 = (1,0,1)$$

The formula for the second vector in the orthogonal basis is $W_2 = v_2 - \frac{(v_2,w_1)}{\|w_1\|^2} w_1$

The quantities that we need for this step are

$$(v_2,w_2)=((1,3,1),(1,0,1))$$

$$=1 + 0 +1 = 2$$

$$\|w_1\|^2=(w_1,w_2)=1^2 +0^2 +1^2 =2$$

The quantities that we need for this step are

$$(v_2,w_2)= ((1,3,1),(1,0,1))$$

$$=1 + 0 +1 = 2$$

$$\|w_1\|^2=(w_1,w_2)=1^2 +0^2 +1^2 =2 .$$

Therefore the second vector is

$w_2 = (1,3,1) - \frac{2}{2}(1,0,1)$

$= (1,3,1) - (1,0,1) \Rightarrow (0,3,0).$

The formula for the third (and final) vector in the orthogonal basis is

$$w_3 = v_3 - \frac{(v_3,w_1)}{\|w_1\|^2} w_1 - \frac{(v_3,w_1)}{\|w_2\|^2} w_2$$

The quantities that we need for this steps are

$\|w_2\|^2 = (w_2,w_2) = 0^2 + 3^2 + 0^2 = 9.$

$(v_3,w_1) = ((3,2,1),(1,0,1)) = 3 + 0 + 1 = 4$

$(v_3,w_2) = ((3,2,1),(0,3,0)) = 0 + 6 + 0 = 6$

Therefore the third vector is

$W_3 = (3,2,1) - \frac{4}{2}(1,0,1) - \frac{6}{9}\ (0,3,0)$

$= (3,2,1) - 2(1,0,1) - \frac{2}{3}(0,3,0)$

$= (1,0,-1).$

$\|w_3\|^2 = (w_3,w_3) = 1^2 + 0^2 + (-1)^2 = 2\ .$

The orthogonal basis is

$\{(1,0,1),(0,3,0),(1,0,-1)\}$

The orthonormal basis is

$$\beta = \{b_1, b_2, b_3\},$$

Where $b_1 = \frac{(w_1)}{\|w_1\|}$, $b_2 = \frac{(w_2)}{\|w_2\|}$, $b_3 = \frac{(w_3)}{\|w_3\|}$,

$\|w_1\|^2 = 2 \Rightarrow \|w_1\| = \sqrt{2}$

$\|w_2\|^2 = 9 \Rightarrow \|w_2\| = 3$

$\|w_3\|^2 = 2 \Rightarrow \|w_3\| = \sqrt{2}$

$$b_1 = \frac{(w_1)}{\|w_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$b_2 = \frac{(w_2)}{\|w_2\|} = (0,1,0)$$

$$b_3 = \frac{(w_3)}{\|w_3\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

Therefore the orthonormal basis is

$$\beta = \{b_1, b_2, b_3\}$$

$$\beta = \{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0) \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\}.$$