NULL SPACES AND RANGES

Null space (or) Kernal

Let V and W be vector spaces and let $T: V \rightarrow W$ be linear transformation Then the set of all vectors x in V such that $T(x) = 0_W$ is called the null space (o kernel) of T. It is denoted by $(T\ )$.

$$\text{(i.e) } N(T) = \{x \in V: T(x) = 0_W\}$$

Note: 0_W is the zero element of W.

Range or Image

Let V and W be vector spaces and let $T: V \rightarrow W$ be linear transformation. Then the subset of W consisting of all images under T of vectors in V is called range o. image of T. It is denoted by $R(T)$

$$\text{(i.e) } R(T) = \{T(x): x \in V\}$$

Theorem 2.1: Let V and W be vector spaces and $T: V \rightarrow W$ be linear. Then

(a) $N(T)$ is a sub space of V and

(b) $R(T)$ is a subspace of W.

Proof: Given that V and W are vector spaces.

$T: V \rightarrow W$ is linear.

(a) To prove $N(T)$ is a subspace of V.

We have to prove for $\alpha, \beta \in F$ and $x, y \in N(T) \Rightarrow \alpha x + \beta y_6$

Since T is linear, $\mu y \in N(\gamma$

$T(0_V) = 0_W; 0_V$ -zero vector of V and 0_W -zero vector of W.

$\therefore 0_V \in N(T)$

$\therefore N(T)$ is non-empty.

Let $x, y \in N(T)$ and $\alpha, \beta \in F$

$\Rightarrow T(x) = 0_W, T(y) = 0_W$

$T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$ ($\because$ Tis linear)

$$=\alpha(0_W)+\beta(0_W)=0_W$$

$\therefore T(\alpha x+\beta y)=0$

$\Rightarrow \alpha x+\beta y\in N(T)$

$\therefore N(T)$ is subspace of V

(b) To prove $R(T)$ is subspace of W.

We have to prove for $\alpha,\beta\in F$ and $x,y\in R(T)\Rightarrow \alpha x+\beta y\in N(T)$

Since $T(0_V)=0_w$ ($\because T$ is linear)

$\Rightarrow 0_w\in R(T)$

$\therefore R(T)$ is non-empty.

Let $x,y\in R(T)$ and $\alpha,\beta\in F$

Then there exits u and v in V such that

$T(u)=x$ and $T(v)=y$

$\alpha x+\beta y\ =\alpha T(u)+\beta T(v)$

$$=T(\alpha u+\beta v)\in R(T)[\because \alpha u+\beta v\in V]$$

$\therefore \alpha x+\beta y\in R(T)$

$\therefore R(T)$ is a subspace of W.

Theorem 2.2: Let V and W be vector spaces over a field F. Let T:V-> linear transformation which is onto. Then T maps a basis of V onto a W.

Proof Let $\{v_1,v_2,\dots,v_n\}$ be a basis for V.

We shall prove that $T(v_1),T(v_2),\dots,T(v_n)$ are linearly independent and that the span W.

Now , $\alpha T(v_1)+\alpha_2T(V_2)+\cdots+\alpha_nT(v_n)=0$

$$\Rightarrow T(\alpha_1v_1)+T(\alpha_2v_2)+\cdots+T(\alpha_nv_n)=0.[\because T \text{ is linear}]$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots . + \alpha_n v_n) = 0$$

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ (since $v_1, v_2, \ldots, v_n$ are linearly independent).

$\therefore T(v_1), T(v_2), \ldots .. T(v_n)$ are linearly independent.

Now, let $w \in W$. then since T is onto, there exists a vector $v \in V$ such $T(v) = w$

Since $L(S) = V$,

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

Then

$$w = T(v)$$

$$= T(\alpha_1 v_1 + \cdots + \alpha_n v_n)$$

$$= \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) [\because T \text{ is linear}]$$

Thus w is a linear combination of the vectors $T(v_1), T(v_2), \ldots, T(v_n)$. $\therefore$ $T(v_1), T(v_2), \ldots, T(v_n)$ span W an hence is a basis for W.

2.1.3. NULLITY AND RANK

Definition

Let V and W be vector spaces, and let $T: V \to W$ be a linear transformation. If $N(T)$ and $R(T)$ are finite-dimensional, then we define

$$\text{nullity } (T) = \dim[N(T)]$$

$$\text{rank}(T) = \dim[R(T)]$$

Note: If nullity $(T) = \{0\}$, then $\dim[N(T)] = 0$ i.e., nullity $(T) = 0$

Theorem 2.5: Rank-Nullity Theorem (or dimensional theorem) Let $T: V \to W$

be a linear transformation and V be a finite dimensional vector space. Then

$$\dim[R(T)] + \dim[N(T)] = \dim(V)$$

$$\text{(i.e) ran } k(T) + \text{nullity } (T) = \dim(V)$$

Proof:

Let V be a vector space of dimension m.i.e., $\dim(V) = m$. Since $N(T)$ is subspace of the finite dimensional vector space V dimension of $N(T)$ is also finite

Let $\dim(N(T)) = n$

Since $N(T)$ is a subspace of $V, n \leq m$

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis $N(T)$. Since $v_i \in N(T)$, for $1 \leq i \leq n$,

Then $T(v_l) = 0, 1 \leq i \leq n$

Since β is a basis of $N(T), \beta$ is linearly independent in $N(T)$.

Therefore β is linearly independent in V.

We shall extend this set β to a basis of the vector space V.

Let this basis of V be $\beta_1 = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_s\}$, where $n + s = m$.

Let $\gamma = \{T(u_1), T(u_2), \dots, T(u_s)\}$.

We shall show that this set γ is a basis of $R(T)$.

ie, to prove $L(\gamma) = R(T)$

and γ is linearly independent.

Since β_1 is a basis of V, it spans V. Hence the set

$\{T(v_1), T(v_2), \dots, T(v_n), T(u_1), T(u_2), \dots, T(u_s)\}$span $R(T)$

Since $T(v_i) = 0$, for $1 \leq i \leq n$

the set $\{T(u_1), T(u_2), \dots, T(u_s)\}$ spans $R(T)$.

$$L(\gamma) = R(T)$$

To prove is linearly independent.

Let $a_1T(u_1) + a_2T(u_2) + \cdots + a_sT(u_s) = 0$.

⇨ $T(a_1u_1 + a_2u_2 + \cdots + a_su_s) = 0$[$\because T$ is linear]

⇨ $a_1u_1 + a_2u_2 + \cdots + a_su_s \in N(T)$

Since $\beta = (v_1, v_2, \ldots, v_n)$ is a basis in $N(T)$,

$a_1u_1 + a_2u_2 + \cdots + a_su_s = b_1v_1 + b_2v_2 + \cdots + b_nv_n$

$\Rightarrow a_1u_1 + a_2u_2 + \cdots + a_su_s - b_1v_1 - b_2v_2 - \cdots - b_nv_n = 0$

Since β_1 is a basis of $V, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_s$ are linearly independent,

$$\therefore a_1 = a_2 = \cdots = a_s = 0, b_1 = b_2 = \cdots = b_n = 0$$

$$\therefore a_1T(u_1) + a_2T(u_2) + \cdots + a_sT(u_s) = 0$$

$$\Rightarrow a_1 = a_2 = \cdots = a_s = 0$$

$\gamma = \{T(u_1), T(u_2), \ldots, T(u_s)\}$ is linearly independent

$\therefore \gamma$ is a basis of $R(T)$

$$\therefore \dim R(T) = s$$

We have $m = n + s$

$$\therefore \dim(V) = \dim[N(T)] + \dim[R(T)]$$

i.e., ran $k(T) +$ nullity $(T) = \dim(V)$

Theorem 2.6: Let V and W be vector space, and let $T: V \to$ W be transformation. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof:

Assume: T is $1 - 1$ (one-to-one)

Let $u \in N(T)$. Then

$$T(u) = 0 = T(0)$$

$$\therefore T(u) = T(0)$$

$$\Rightarrow u = 0 \; (\because T \text{ is } 1-1)$$

$$\therefore N(T) = \{0\}$$

Conversely, assume that $N(T) = \{0\}$

Let $T(u) = T(v)$.

$$T(u) - T(v) = 0$$

$$T(u - v) = 0 \; (\because T \text{ is linear })$$

$$\Rightarrow u - v \in N(T) = \{0\}$$

$$\therefore u - v = 0$$

$$\therefore u = v$$

$\therefore T$ is $1-1$ (one-to-one).

Theorem 2.7: Ift V and W be finite dimensional over F and $T: V \rightarrow W$ be Then the following are equivalent.

1 T is one-to-one

2 T is onto

3 rank $(T) = \dim(W)$

Proof:

By dimensional theorem we have

$\operatorname{rank}(T) + \text{nullity } (T) = \dim(V) \ldots ($

T is one-to-one $\Leftrightarrow N(T) = \{0\}$

$\Leftrightarrow$ nullity $(T) = 0$

$\Leftrightarrow \operatorname{rank}(T) = \dim V$ [$using$ (1)]

$\Leftrightarrow \dim R(T) = \dim W$ [T is $1-1$]

$\Leftrightarrow R(T) = W[\because R(T) \subseteq W$ with same rank]

$\Leftrightarrow T'$ is onto.

2. 1.4. PROBLEMS UNDER RANK AND NULLITY

Let V and W be vector space and let $T: V \to W$ be linear map

- $N(T) = \{x \in V: T(x) = 0_W\}$
- nullity $(T) = \dim(N(T))$
- $R(T) = \{T(x): x \in V\}$
- $\operatorname{rank}(T) = \operatorname{image}(R(T))$

Example 17. Let $T: R^3 - R^2$ by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. Find N and $R(T)$

Sol: To find $N(T)$:

$N(T) = \{(a_1, a_2, a_3) \in R^3: T(a_1, a_2, a_3) = 0\}$

Let $T(a_1, a_2, a_3) = 0$

$(a_1 - a_2, 2a_3) = 0$

Equating each terms to zero, we get

$$2a_3 = 0$$

$$a_3 = 0$$

$$a_1 - a_2 = 0$$

$$a_1 = a_2$$

$\mathrm{N}(T) = \{(a_1, a_2, a_3)\}$

$=\{(a_1, a_1, 0): a_1 \in R\}$

To find $R(T)$:

The usual basis of R^3 is $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$

Given, $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$$T(1,0,0) = (1,0)$$

$$T(0,1,0) = (-1,0) = -1(1,0)$$

$$T(0,0,1) = (0,2) = 2(0,1)$$

$$\text{lmage}(T) = \text{span}\{(1,0), -(1,0), 2(1,0)\}$$

$$= \text{span}\{(1,0),(1,0)\} \begin{bmatrix} * -(1,0) \text{ is depending on } (1,0 \\ 2(1,0) \text{ is multiple of } (1,0) \end{bmatrix}$$

$$= \{x(1,0) + y((1,0)\}$$

$$= \{(x, y)\}$$

$$= R^2$$

Example 18. Let $T: R^2 \to R^3$ be a linear map defined by $T(a_1, a_2) = (a_1 - a_2, 0,0)$. Find nullity (T) and rank(T).

Sol: To find nullity (T) :

$N(T) = \{(a_1, a_2) \in R^2 : T(a_1, a_2) = 0\}$

Let $T(a_1, a_2) = 0$

$$(a_1 - a_2, 0,0) = (0,0,0)$$

$$a_1 - a_2 = 0$$

$$a_1 = a_2$$

$N(T) = \{(a_1, a_1)/a_1 \in R\}$

$$=\{(1,1)a_1/a_1 \in R\}$$

The basis of $N(T)$ is $\beta = \{(1,1)\}$

The nullity of $T = \dim[N(T)] = 1$

To find range (T) :

The usual basis of R^2 is $\beta = \{(1,0),(0,1)\}$

Given, $T(a_1, a_2) = (a_1 - a_2, 0,0)$.

$$T(1,0) = (1,0,0)$$

$$T(0,1) = (-1,0,0)$$

The image of usual basis span Image (T)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to R_2 + R_1$

The basis of $R(T)$ is the non-zero row of the echelon matrix.

Therefore the basis of $R(T)$ is $\gamma = \{(1,0,0)\}$.

$\text{rank}(T) = \dim[R(T)] = 1$

Example (19) Let $T: R^2 \to R^3$ be the linear mapping defined by $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2, a_2)$

Find the basis and dimension of (a) null space of T (b) Range of T

Sol: (a) To find (null space) kernel of T :

Let $T(a_1, a_2) = 0$

$(a_1 + a_2, a_1 - a_2, a_2) = (0,0,0)$

Equating the like terms

$$a_1 + a_2 = 0 \ldots (1)$$

$$a_1 - a_2 = 0 \ldots (2)$$

$$a_2 = 0$$

$(1) \Rightarrow a_2 = 0$

kernel of $T = N(T) = \{(0,0)\}$

The nullity of $T = \dim(N(T)) = 0$

(b) To find range of T :

The usual basis of R^2 is $\beta = \{(1,0),(0,1)\}$

Given, $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2, a_2)$

$$T(1,0) = (1,1,0)$$

$$T(0,1) = (1,-1,0)$$

The image of usual basis span Image(T)

$$Let A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} R_2 \to R_2 - R_1$$

The basis of $R(T)$ is the non-zero row of the echelon matrix

Thus $y = \{(1,0,1),(0,-2,0)\}$ forms a basis for $\text{Im}(T)$.

Hence $\dim[\text{lm}(T)] = 2$

i.e., Rank $(T) = 2$

Example 20. Let $T: R^3 \to R^3$ be the linear mapping defined b $T(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3)$

Find the basis and dimension of (a) Kernel(b) Image of T

Sol: (a) To find kernel of T :

Let $T(a_1, a_2, a_3) = 0$

$$(a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3) = (0,0,0)$$

$$a_1 + 2a_2 - a_3 = 0 \dots \ (1)$$

$$a_2 + a_3 = 0 \dots \ (2)$$

$$a_1 + a_2 - 2a_3 = 0 \dots \ (3)$$

Solve $(1), (2) \& (3)$

The matrix of the given equations is

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} R_3 \to R_3 + R_2$$

$$a_1 + 2a_2 - a_3 = 0 \dots (4)$$

$$a_2 + a_3 = 0 \dots \ (5)$$

Adding (4) and (5), we get

$$a_1 + 3a_2 = 0$$

$$a_1 = -3a_2 \dots (6)$$

a_1 is depending on a_2.

Therefore the basis of $N(T)$ contains one element

$$(6) \Rightarrow \quad \frac{a_2}{-3} = \frac{a_2}{1}$$

$$a_1 = -3, a_2 = 1$$

$(5) \Rightarrow a_3 = -1$

Basis of kernel of T is $\beta = \{(-3,1,-1)\}$

$\text{nulliy}(T) = \dim(N(T)) = 1$

(b) To find Image (T) :

The basis of R^3 is $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$

Given, $T(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3)$

$$T(1,0,0) = (1,0,1)$$

$$T(0,1,0) = (2,1,1)$$

$$T(0,0,1) = (-1,1,-2)$$

The image of usual basis span Image (T)

Let $\quad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2.$$

The basis of $R(T)$ is the non-zero row of the echelon matrix

Thus $\gamma = \{(1,0,1),(0,1,-1)\}$ form a basis form $\text{Im}(T)$.

Hence $\dim[\text{Im}(T)] = 2$

i.e, $\text{Rank}(T) = 2$

Example 21. Let $T: P_3(R) \to P_2(R)$ defined by $T[f(x)] = f'(x)$. find the nullity and rank of T.

Sol: Let $f(x) \in P_3(x)$. Then

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \dots$ (1)

$f'(x) = a_1 + a_2x + a_3x^2$

To find nullity (T) :

$N(T) = \{f(x) \in P_3(R): T(f(x)) = 0\} \dots (2)$

Let $T(f(x)) = 0$. Then

$F'(x) = 0$

$a_1 + a_2x + a_3x^2 = 0$

$a_1 + a_2x + a_3x^2 = 0+0x+0x^2$

$a_1 = 0, a_2 = 0, a_3 = 0$

Example 23. Find the range space, kernel, rank and nullity of the following linear transformation. Also verify the rank-nullity theorem $T: V_2(R) \to V_2(R)$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$

Sol: To find (T) :

Let $T(x_1, x_2) = 0$

$(x_1 + x_2, x_1) = 0$

$x_1 + x_2 = 0 \dots (1)$

$x_1 = 0$

$(1) \Rightarrow x_2 = 0$

$\therefore N(T)$ contain only zero element of $V_2(R)$.

$\therefore N(T) = \{(0,0)\}$

i.e the null space = $\{(0,0)\}$

$\dim[N(T)] = 0$

i.e nullity $= 0$

To find $R(T)$:

The standard basis $e_1 = (1,0), e_2 = (0,1)$ of $V_2(R)$

$T(e_1) = T(1,0)$

$= (1 + 0,1)$

$T(e_2) = T(0,1)$

$= (0 + 1,0)$

$= (1,0)$

The image of usual basis span Image (T)

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} R_2 \rightarrow R_2 - R_1$

This is in the echelon form there are two non-zero rows

Basis of Image (T) is $\gamma = \{(1,1), (0, -1)\}$

Therefore, rank of $T = 2$

Hence R(T) is the subspace generated by $(1,1)$ and $(0, -1)$

$R(T) = x_1(1,1) + x_2(0, -1)$

$= (x_1, x_1) + (0, -x_2)$

$= (x_1, x_1 - x_2)$ for all $x_1, x_2 \in R$

i.e the range space $= \{x_1, x_1 - x_2\} = V_2(R)$

Rank $+$ nullity $= 2 + 0$

$= 2$

$= \dim[V_2(R)]$

Hence the nullity theorem is verified.

Example(24) Let $T: R^3 \to R^3$ defined by Y(x,y,z)=(x+y,x-y,2x+z). Find the range space, null-space, rank and nullity of T and verify rank+nullity of T = $\dim(\mathrm{R}^3)$.

Sol: To find (T) :

$T(x,y,z) = 0(x, x-2y, 2x, 2x-2y+z) = (0,0,0)$

$x_1 + x_2 = 0$ ... (1)

$x_1 - x_2 = 0$ ... (2)

$2x_1 + x_3 = 0$ ... (3)

$(1) + (2) \Rightarrow 2x_1 = 0$

$x_1 = 0$

(1) $\Rightarrow x_2 = 0$

(3) $\Rightarrow x_3 = 0$

$\therefore N(T) = \{(0,0,0)\}$

$= \{0\}$

$\Rightarrow \dim[N(T)] = 0$

$\Rightarrow$nullity $= 0$

To find $R(T)$:

The standard basis of R^3 is $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

Let $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$

$T(x,y,z) = (x+y, x-y, 2x+z)$

$T(1,0,0) = (1+0, 1-0, 0+0)$

$= (1,1,0)$

$T(0,0,0) = (0+1, 0-1, 0+0)$

$= (1,-1,0)$

$T(0,0,1) = (0+0, 0-0, 0+1)$

$= (0,0,1)$

The image of usual basis span Image (T)

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This in echelon from and there are three non-zero rows.

$\dim[R(T)] = 3$

i.e rank of $T = 3$

$R(T) =$ the subspace generated by $(1,1,0), (0,-2,0), (0,0,1)$

$= x(1,1,2) + y(0,-2,-2) + z(0,0,1)$

$= (x, x, 2x) + (0, -2y, -2y) + (0,0,z)$

$R(T) = (x, x-2y, 2x, 2x-2y+z)$

Rank + nullity $= 3 = \dim(R^3)$

Fxample 25. Find the range space, kernel, rank and nullity of the following linear transformation. Also verify the rank-nullity theorem defined by

T: $V_3(R) \rightarrow V_2(R) by$

$T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$

Sol: To find the kernel,

Since the linear transformation is not given, first find the linear transformation

The usual basis of R^3 is $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

$e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$

Given $T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$

$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$= x_1e_1 + x_2e_2 + x_3e_3$

$T(x_1, x_2, x_3) = T(x_1e_1 + x_2e_2 + x_3e_3)$

$= x_1T(e_1) + x_2T(e_2) + x_3T(e_3)$

$= x_1(2,1) + x_2(0,1) + x_3(1,1)$

$= (2x_1, x_1) + (0, x_2) + (x_3, x_3)$

$= (2x_1 + x_3, x_1 + x_2 + x_3)$

$N(T) = \{(x_1, x_2, x_3): T(x_1, x_2, x_3) = 0\}$

Put $T(x_1, x_2, x_3) = 0$

$(2x_1 + x_3, x_1 + x_2 + x_3) = (0,0)$

$2x_1 + x_3 = 0 \ldots (1)$

$x_1 + x_2 + x_3 = 0$

$(1) \Rightarrow x_3 = -2x_1$

$(2) \Rightarrow x_1 + x_2 - 2x_1 = 0$

$x_2 - x_1 = 0$

$x_1 = x_2$

$\frac{x_1}{1} = \frac{x_2}{1}$

$(3) \Rightarrow x_3 = -2$

The basis of $N(T)$ is $\beta = \{(1,1,-2)\}$

$N(T) =$ the subspace generated by $(1,1,-2)$

$=\{(1,1,-2)x_1\}$

$=\{(x_1, x_1, -2x_1)\}$

$\therefore \dim[N(T)] = 1$

i.e nullity $(T) = 1$

To find $\dim[R(T)]$

Given $T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$

The image of usual basis span Image (T).

$$\text{Let } A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} R_1 \leftrightarrow R_3$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} R_3 \rightarrow R_3 - 2R_1$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} R_3 \rightarrow R_3 + R_2$$

This is the echelon form and there are 2 non-zero rows in it.

The basis of $R(T)$ is $\gamma = \{(1,1), (0,1)\}$

$\therefore \dim[R(T)] = 2$

i.e rank of $T = 2$

Range space = the subspace generated by (1,1) and (0,1)

$= x_1(1,1) + x_2(0,1)$

$= (x_1, x_1) + (0, x_2)$

$= (x_1, x_1 + x_2)$

$\therefore$ Range space $= \{(x_1, x_1 + x_2) \mid x_1, x_2 \in R\}$

Rank(T) + nullity $(T) = 2 + 1$

$= 3$

$= \dim(R^3)$