DEPARTMENT OF MATHEMATICS

BIG – M – METHOD

INTRODUCTION

The Big M method is a version of the Simplex Algorithm that first finds a best feasible solution by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm. The iterative procedure of the algorithm is given below.

Step-1 : Modify the constraints so that the RHS of each constraint is non-negative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if any negative number multiplies an inequality, the direction of the inequality is reversed). After modification, identify each constraint as a <, >, or = constraint.

Step-2 : Convert each inequality constraint to standard form (If a constraint is a ≤ constraint, then add a slack variable Xi; and if any constraint is a ≥ constraint, then subtract an excess variable Xi , known as surplus variable).

Step-3: Add an artificial variable a1 to the constraints identified as '≥' or with '=' constraints at the end of Step2. Also add the sign restriction $a_i \geq 0$.

Step-4: Let M denote a very large positive number. If the LP is a minimization problem, add (for each artificial variable) Mai to the objective function. If the LP is a maximization problem, add (for each artificial variable) -Mai to the objective function.

Step-5: Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex.

Problem :1

Solve the following LPP by using Big -M Method

Maximize Z = 6X1+4X2

Subject to constraints:

2X1+3X2<=30, 3X1+2X2<=24, X1+X2>=3

Solution

Introducing slack variables S1>=0, S2>=0 to the first and second equations in order to convert <= type to equality and add surplus variable to the third equation S3 >=0 to convert >= type to equality. Then the standard form of LPP is

MAX Z=6X1+4X2+0S1+0S2+0S3

Subject to constraints

2X1+3X2+S1=30, 3X1+2X2+S2=24, X1+X2-S3=3

Clearly there is no initial basic feasible solution. So an artificial variable A1>=0 is added in the third equation. Now the standard form will be

MAX Z=6X1+4X2+0S1+0S2+0S3+A1

Subject to constraints

2X1+3X2+S1=30, 3X1+2X2+S2=24

X1+X2-S3+A1=3

			6	4	0	0	0	-M	
Cb	YBa	Xba	Y1	Y2	Y3	Y4	Y5	Y6	Xbi/ Yi1
0	Y3	X3=30	2	3	1	0	0	0	30/2= 15
0	Y4	X4= 24	3	2	0	1	0	0	24/3= 8
-M	Y6	A1=3	1	1	0	0	-1	1	3/1= 3
	Zj-Cj		-M-6	-M-4	0	0	-1	1	

It clearly shows the net evaluations for each column variable; from these calculations, it is clear that X1 is the entering variable and A1, the artificial variable leaves the basis. Introduce Y1 and drop Y6. Then, in the usual row operations, we modify this table and the new table is arrived.

			6	4	0	0	0	-M	Xbi/Yi1
Cb	YBa	Xba	Y1	Y2	Y3	Y4	Y5	Y6	
0	Y3	X3=24	0	1	1	0	2	-2	24/2= 12
0	Y4	X4=15	0	-1	0	1	3	-3	15/3= 5
6	Y1	X1= 3	1	1	0	0	-1	1	-
Zj-Cj			0	2	0	0	-6	6	

Introduce Y5 and drop Y4 from the basis; the new modified **table-3** is given below;

			6	4	0	0	0	-M	Xbi/i1
Cb	YBa	Xba	Y1	Y2	Y3	Y4	Y5	Y6	
0	Y3	X3=14	0	5/3	1	-2/3	0	0	
0	Y5	X5=5	0	-1/3	0	1/3	1	-1	
6	Y1	X1= 8	1	2/3	0	1/3	0	0	
Zj-Cj			0	0	0	2	0	0	

Since all Zj –Cj >=0, an optimum solution has reached. Thus an optimum basic feasible solution to the LPP is X1=8, X2=0, MAX Z =48

Problem : 2

Use big M method to solve a given LPP. Minimize Z = 5X1-6X2-7X3

Subject to constraints

X1+5X2-3X3>=15, 5X1-6X2+10X3<=20, X1+X2+X3=5

X1, X2 & X3 >= 0

Solution

Introducing slack variables X4>=0 to the first equations in order to convert <= type to equality and add surplus variable X5>=0 to the second equation in order to convert >= type to equality.

Then the standard form of LPP is

MIN Z=5X1-6X2-7X3+X4-X5

Subject to constraints

5X1-6X2+10X3+X4=20, X1+5X2-3X3-X5=15, X1+X2+X3=5

Clearly there is no initial basic feasible solution. So two artificial variables A1>=0 and A2>=0 are added in the second and third equation. Now the standard form will be

MIN Z=5X1-6X2-7X3+X4-X5+A1+A2

Subject to constraints

5X1-6X2+10X3+X4=20, X1+5X2-3X3-X5+A1=15, X1+X2+X3+A2=5

Simplex Table-1

			5	-6	-7	0	0	-M	-M	Xbi/
Cb	**YBa**	**Xba**	**Y1**	**Y2**	**Y3**	**Y4**	**Y5**	**Y6**	**Y7**	**Yi1**
0	Y4	X4=20	5	-6	10	1	0	0	0	-
M	Y6	X6=15	1	5	-3	0	-1	1	0	15/5 = 3
M	Y7	X7= 5	1	1	1	0	0	0	1	5/1=5
Zj-Cj			2M-5	6M+6	2M+7	0	-M	0	0	

From the calculations related to entering column and leaving variable, which is summarized in the above table, it is clear that introduces X2 and drop X6 from the basis. The new simplex table is given below.

Simplex Table-2

			5	-6	-7	0	0	M	M	Xbi/ Yi1
Cb	YBa	Xba	Y1	Y2	Y3	Y4	Y5	Y6	Y7	
0	Y4	X4=38	31/5	0	32/5	1	-6/5	6/5	0	190/31
-6	Y2	X6=3	1/5	1	-3/5	0	-1/5	1/5	0	-
M	Y7	X7= 2	4/5	0	8/5	0	1/5	-1/5	1	10/8
Zj-Cj			(-31+ 4M)/5	0	3+8 M/5	0	6+ M/5	-	-	

Now, we see that we have to include X3 and drop X7 from the basis.

The new modified table- 3 is given below

Simplex Table-3

			5	-6	-7	0	0	M	M	Xbi/Yi1
Cb	YBa	Xba	Y1	Y2	Y3	Y4	Y5	Y6	Y7	
0	Y4	X4=30	31/5	0	0	1	-2	2	-4	
-6	Y2	X2=15/4	1/5	1	0	0	-1/8	1/8	3/8	
-7	Y3	X3= 5/4	0	0	1	0	-1/8	1/8	5/8	
Zj-Cj			-31/5	0	0	0	-1/8	-	-	

Since all Zj –Cj <=0, an optimum solution has reached. Thus an optimum basic feasible solution to the LPP is X3=5/4, X2=15/4,

MIN Z= (5*0) + (-6*15/4) + (-7* 5/4) = (-90/4) + (-35/4) = - 125/4

Problem : 3

Use penalty (or Big 'M') method to

Minimize $z = 4x_i + 3x_2$

subject to the constraints :

$2x_1 + x_2 \geq 10, \quad -3x_1, + 2x_2 \leq 6$

$x_1 + x_2 \geq 6, \quad x_1 \geq 0$ and $x_2 \geq 0$.

Solution. Introducing surplus (negative slack) variables $x_3 \geq 0$, $x_5 \geq 0$ and slack variable $x_4 \geq 0$ in the constraint inequations, the problem becomes

Maximize $z^* = -4x_1 - 3x_2 + 0.x_3 + 0.x_4 + 0.x_5$

subject to the constraints :

$$2x_1 + x_2 - x_3 = 10, \quad -3x_1 + 2x_2 + x_4 = 6$$

$$x_l + x_2 - x_5 = 6, \quad x_j \geq 0 \; Q(j = 1,2, 3, 4, 5)$$

Clearly, we do not have a ready basic feasible solution. The surplus variables carry negative coefficients (-1). We introduce two new variables $A_1 \geq 0$ and $A_2 \geq 0$ in the first and third equations respectively. These extraneous variables, commonly termed as artificial variables, play the same role as that of slack variables in providing a starting basic feasible solution.

We assign a very high penalty cost (say $-M$, $M \geq 0$) to these variables in the objective function so that they may be driven to zero while reaching optimality.

Now the following initial basic feasible solution is available :

$$A_l = 10, x_4 = 6 \quad \text{and } A_2 = 6$$

with $\mathbf{B} = (\mathbf{a_6}, \mathbf{a_4}, \mathbf{a_7})$ as the basis matrix. The cost matrix corresponding to basic feasible solution is $\mathbf{c_B} = (-M, 0, -M)$

Now, corresponding to the basic variables A_1, x_4 and A_2. the matrix $\mathbf{Y = B^{-1}A}$ and the net evaluations $z_j - c_j$ (j = 1, 2, ... 7) are computed. The initial basic feasible solution is displayed in the following simplex table :

Initial Iteration.			Introduce y_1 and drop y_6.						
			—4	—3	0	0	0	-M	-M
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
$-M$	y_6	10	2*	1	-1	0	0	1	0
0	y_4	6	-3	2	0	1	0	0	0
$-M$	y_7	6	1	1	0	0	- 1	0	1
	z*	-16M	-3W+4	-2M+3	M	0	M	0	0

We observe that the most negative zj - cj is 4 - 3M (= $z_l - c_1$). The corresponding column vector y_1, therefore, enters the basis. Further, since min. = 5; the element y_{11} (=2) becomes the leading element for the first iteration

First Iteration: Introduce y_2 and drop y_7.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_7
-4	y_1	5	1	1/2	-1/2	0	0	0
0	y_4	21	0	7/2	-3/2	1	0	0
-M	y_7	0	0	1/2*	1/2	0	-1	0
	z*	-M-20	0	$\frac{-M}{2}+1$	$\frac{-M}{2}+2$	0	M	0

In the above table, we omitted all entries of column vector $\mathbf{y_6}$, because the artificial variables A_l has left the basis and we would not like it to re-enter in any subsequent iterations.

Now since the most negative (z_j—c_j) is z_2-c_2; the non-basic vector $\mathbf{y_2}$ enters the basis. Further, since min   is 2 which occurs for the element y_{32} (= 1/2), the corresponding basis vector $\mathbf{y_7}$ leaves the basis and the element y_{32} becomes the leading element for the next iteration.

Final Iteration: Optimum Solution,

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
-4	y_1	4	1	0	-1	0	1
0	y_4	14	0	0	-5	1	7
-3	y_2	2	0	1	1	0	-2
	z^*	-22	0	0	1	0	2

It is clear from the table that all $z_j - c_j$ are positive. Therefore an optimum basic feasible solution has been attained which is given by

$x_1 = 4$, $x_2 = 2$, maximum $z = 22$.

Problem : 4

Maximize $z = 3x_1, + 2x_2$ subject to the constraints :

$$\mathbf{2x_1 + x_2 \le 2,\ \ 3x_1 + 4x_2 \ \ge\ 12,\ x_1, x_2\ \ge 0.}$$

Solution:

Introducing slack variable $x_3 \ge 0$, surplus variable $x_5 \ge 0$ and an artificial variable $A_1 \ge 0$, the reformulated L.P.P. can be written as :

Maximize $z = 3x_1 + 2x_2 + 0.x_3 + 0.x_4 - MA_1$

subject to the constraints :

$2x_1 + x_2 + x_3 = 2,$

$3x_1 + 4x_2 - x_4 + A_1 = 12$

$x_1, x_2, x_3, x_4 \geq 0$ and $A_1 \geq 0.$

starting basic feasible solution is :

x3 = 2 and $A_1 = 12.$

The iterative simplex tables are :

Initial Iteration: Introduce y_2 and drop y_3.

			3	2	0	0	-M
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	2	2	1*	1	0	0
-M	y_5	12	3	4	0	-1	1
	z	-12M-2	-3M-3	-4M	0	M	0

Final Iteration. No solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
2	y_2	2	2	1	1	0	0
-M	y_5	4	-5	0	-4	-1	1
	z	4M+4	5M+1	0	4M + 2	M	0

Here the coefficient of *M* in each $z_j - c_j$ is non-negative and an artificial vector appears in the basis, not at the zero level. Thus the given L.P.P. does not possess any feasible solution.

Problem 5
Max $Z = -2x_1 - x_2$
Subject to

$3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Max $Z = -2x_1 - x_2 + 0s_1 + 0s_2 - M a_1 - M a_2$
Subject to

$3x_1 + x_2 + a_1 = 3$
$4x_1 + 3x_2 - s_1 + a_2 = 6$
$x_1 + 2x_2 + s_2 = 4$
$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

	$C_j \rightarrow$		-2	-1	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	3	$\boxed{3}$	1	0	0	1	0	3 /3 = 1→
a_2	-M	6	4	3	-1	0	0	1	6 / 4 =1.5
s_2	0	4	1	2	0	1	0	0	4 / 1 = 4
	Z = -9M		↑ 2 – 7M	1 – 4M	M	0	0	0	←Δ_j
x_1	-2	1	1	1/3	0	0	X	0	1/1/3 =3
a_2	-M	2	0	$\boxed{5/3}$	-1	0	X	1	6/5/3 =1.2→
s_2	0	3	0	5/3	0	1	X	0	4/5/3=1.8
	Z = -2 – 2M		0	↑ $\frac{(-5M+1)}{3}$	0	0	X	0	←Δ_j
x_1	-2	3/5	1	0	1/5	0	X	X	
x_2	-1	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	Z = -12/5		0	0	1/5	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = -12/5$, $x_1 = 3/5$, $x_2 = 6/5$

Problem : 6

Max $Z = 3x_1 - x_2$
Subject to

$2x_1 + x_2 \geq 2$, $x_1 + 3x_2 \leq 3$, $x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Max $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - M a_1$ Subject to

$2x_1 + x_2 - s_1 + a_1 = 2$
$x_1 + 3x_2 + s_2 = 3$
$x_2 + s_3 = 4$
$x_1 , x_2 , s_1, s_2, s_3, a_1 \geq 0$

		$C_j \rightarrow$	3	-1	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-M	2	$\boxed{2}$	1	-1	0	0	1	$2 / 2 = 1 \rightarrow$
s_2	0	3	1	3	0	1	0	0	$3 / 1 = 3$
s_3	0	4	0	1	0	0	1	0	-
	Z = -2M		$\uparrow$ -2M-3	-M+1	M	0	0	0	$\leftarrow \Delta_j$
x_1	3	1	1	1/2	-1/2	0	0	X	-
s_2	0	2	0	5/2	$\boxed{1/2}$	1	0	X	$2/1/2 = 4 \rightarrow$
s_3	0	4	0	1	0	0	1	X	-
	Z = 3		0	5/2	$\uparrow$ -3/2	0	0	X	$\leftarrow \Delta_j$
x_1	3	3	1	3	0	1/2	0	X	
s_1	0	4	0	5	1	2	0	X	
s_3	0	4	0	1	0	0	1	X	
	Z = 9		0	10	0	3/2	0	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 9, $x_1 = 3$, $x_2 = 0$

Problem ; 7

Max $Z = 3x_1 + 2x_2 + x_3$

Subject to

$2x_1 + x_2 + x_3 = 12$

$3x_1 + 4x_2 = 11$

and x_1 is unrestricted

$x_2 \geq 0$, $x_3 \geq 0$

Solution

Max $Z = 3(x_1' - x_1'') + 2x_2 + x_3 - M a_1 - M a_2$

Subject to

$2(x_1' - x_1'') + x_2 + x_3 + a_1 = 12$

$3(x_1' - x_1'') + 4x_2 + a_2 = 11$

$x_1', x_1'', x_2, x_3, a_1, a_2 \geq 0$

$\text{Max } Z = 3x_1' - 3x_1'' + 2x_2 + x_3 - M a_1 - M a_2$

Subject to

$$2x_1' - 2x_1'' + x_2 + x_3 + a_1 = 12$$
$$3x_1' - 3x_1'' + 4x_2 + a_2 = 11$$
$$x_1', x_1'', x_2, x_3, a_1, a_2 \ge 0$$

		$C_j \rightarrow$	3	-3	2	1	-M	-M	
Basic Variables	C_B	X_B	X_1'	X_1''	X_2	X_3	A_1	A_2	Min ratio X_B / X_k
a_1	-M	12	2	-2	1	1	1	0	12 /2 = 6
a_2	-M	11	$\boxed{3}$	-3	4	0	0	1	11/3 =3.6→
		Z = -23M	↑ -5M-3	5M+3	-5M-2	-M-1	0	0	←Δ_j
a_1	-M	14/3	0	0	-5/3	$\boxed{1}$	1	X	14/3/1 = 14/3→
x_1'	3	11/3	1	-1	4/3	0	0	X	-
		$Z = \frac{-14M+11}{3}$	0	-6	5/3M+1	↑ -M-1	0	X	←Δ_j
x_3	1	14/3	0	0	-5/3	1	X	X	
x_1'	3	11/3	1	-1	4/3	0	X	X	
		Z = 47/3	0	0	1/3	0	X	X	

Since all $\Delta_j \ge 0$, optimal basic feasible solution is obtained

$x_1' = 11/3$, $x_1'' = 0$

$x_1 = x_1' - x_1'' = 11/3 – 0 = 11/3$

Therefore the solution is Max $Z = 47/3$, $x_1 = 11/3$, $x_2 = 0$, $x_3 = 14/3$