UNIT II – Probability and Random Variables

Random Experiment:

An experiment whose output is uncertain even though all the outcomes are known.

Example: Tossing a coin, Throwing a fair die, Birth of a baby.

Sample Space:

The set of all possible outcomes in a random experiment. It is denoted by S.

Example:

For tossing a fair coin, $S = \{H, T\}$

For throwing a fair die, $S = \{1, 2, 3, 4, 5, 6, \}$

For birth of a baby, $S = \{M, F\}$

Event:

A subset of sample space is event. It is denoted by A.

Mutually Exclusive Events:

Two events A and B are said to be mutually exclusive events if they do not occur simultaneously. If A and B are mutually exclusive, then $A \cap B = \Phi$

Example:

Tossing two unbiased coins

$$S = \{HH, HT, TH, TT\}$$

(i) Let $A = \{HH\}, B = \{HT\}$

$$A \cap B = \{H\} \neq \Phi$$

Then A and B are not mutually exclusive.

(i) Let $A = \{HH\}, B = \{TT\}$

$$A \cap B = \Phi$$

Then A and B are mutually exclusive.

1.1 Probability:

Probability of an event A is $P(A) = \frac{n(A)}{n(S)}$

i.e., $P(A) = \frac{number\ of\ cases\ favourable\ to\ A}{Total\ number\ of\ cases}$

Axioms of Probability:

(i) $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$

(iii) $P(A \cup B) = P(A) + P(B)$, if A and B are mutually exclusive.

Note:

(i) $P(\phi) = 0$

(ii) $P(\overline{A}) = 1 - P(A)$, for any event A

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, for any two events A and B.

Independent events:

Two events A and B are said to be independent if occurrence of A does not affect the occurrence of B.

Condition for two events and B are independent:

$$P(A \cap B) = P(A)\, P(B)$$

Conditional Probability:

If the probability of the event A provided the event B has already occurred is called the conditional probability and is defined as

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

The probability of an event B provided A has occurred already is given by

$$P(B|A) = \frac{P(A\cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

RANDOM VARIABLE

Define Random Variables:

A random variable is a function that assign a real number for all the outcomes in the sample space of a random experiment.

Example:

Toss two coins then the sample space $S = \{HH, HT, TH, TT\}$

Now we define a random variable X to denote the number of heads in 2 tosses.

X(HH) = 2

X(HT) = 1

X(TH) = 1

X(TT) = 0

Types of Random Variables:

(i) Discrete Random Variables
(ii) Continuous Random Variables

DISCRETE RANDOM VARIABLE

Definition : A discrete random variable is a R.V.X whose possible values consitute finite set of values or countably infinite set of values.

Probability mass function (PMF):

Let X be discrete random variable. Then $P(X = x_i) = p(x_i) = p_i$is said to be a Probability mass function of X, if

(i) $0 \leq p(x_i) \leq 1$

(ii) $\sum_i p(x_i) = 1$

The collection of pairs $\{x_i, p_i\}, i = 1, 2, 3, \ldots$ is called the probability distribution of the random variable X, which is sometimes in the form of a table as given below:

$X = x_i$	x_1	x_2	$\cdots$	x_r	$\cdots$
P$(X = x_i)$	p_1	p_2	$\cdots$	p_r	$\cdots$

CUMULATIVE DISTRIBUTION FUNCTION

Let X be a R.V . The CDF of X is $F(x) = P(X \leq x) = \sum_{X \leq x} p(x)$

Note : $p(x_i) = P(X = x_i) = F(x_i) - F(x_{i-1})$, Where F is the distribution function of the random variable X.

Problems on Discrete Random Variables

1. **A Discrete Random Variable X has the following probability distribution**

X	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
P(x)	**a**	**3a**	**5a**	**7a**	**9a**	**11a**	**13a**	**15a**	**17a**

(i) **Find the value of "a".**

(ii) **Find $\boldsymbol{P[X < 3], P[0 < X < 3], P\ [X \geq 3]}$**

(iii) **Find the distribution of X.**

Solution:

(i) We know that $\sum P(x) = 1$

$$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$\Rightarrow 81a = 1$$

$$\Rightarrow a = \frac{1}{81}$$

(ii) $P[X < 3] = P[X = 0] + P[X = 1] + P[X = 2]$

$$= a + 3a + 5a$$

$$= 9a$$

$$= \frac{9}{81}$$

$$P[0 < X < 3] = P[X = 1] + P[X = 2]$$

$$= 3a + 5a$$

$$= 8a$$

$$= \frac{8}{81}$$

$$P\ [X \geq 3] = 1 - P[X < 3]$$

$$= 1 - \frac{9}{81} = \frac{72}{81}$$

(iii) **Distribution of X:**

X	**P(x)**	**F(X) = P[X≤ x]**
0	a	F(0) = P[X≤ 0] = $\frac{1}{81}$
1	3a	F(1) = P[X≤ 1] = $F(0) + P(1) = \frac{1}{81} + \frac{3}{81} = \frac{4}{81}$
2	5a	F(2) = P[X≤ 2] = $F(1) + P(2) = \frac{4}{81} + \frac{5}{81} = \frac{9}{81}$
3	7a	F(3) = P[X≤ 3] = $F(2) + P(3) = \frac{9}{81} + \frac{7}{81} = \frac{16}{81}$
4	9a	F(4) = P[X≤ 4] = $F(3) + P(4) = \frac{16}{81} + \frac{9}{81} = \frac{25}{81}$
5	11a	F(5) = P[X≤ 5] = $F(4) + P(5) = \frac{25}{81} + \frac{11}{81} = \frac{36}{81}$
6	13a	F(6) = P[X≤ 6] = $F(5) + P(6) = \frac{36}{81} + \frac{13}{81} = \frac{49}{81}$
7	15a	F(7) = P[X≤ 7] = $F(6) + P(7) = \frac{49}{81} + \frac{15}{81} = \frac{64}{81}$
8	17a	F(8) = P[X≤ 8] = $F(7) + P(8) = \frac{64}{81} + \frac{17}{81} = \frac{81}{81}$

2. A Discrete Random Variable X has the following probability distribution

X	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
P(x)	**0**	**k**	**2k**	**2k**	**3k**	k^2	$2k^2$	$7k^2 + k$

(i) Find the value of "k".

(ii) Find $P[X < 6], P[1 < X < 5], P\ [X \geq 6], P[X > 2]$

(iii) **Find $P[1.5 < X < 4.5 \,/\, X > 2]$**

(iv) **Find the distribution of X and find the value of k if $P[X < k] > \frac{1}{2}$**

Solution:

(i) We know that $\sum P(x) = 1$

$$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow 10k^2 + 9k = 1$$

$$\Rightarrow 10k^2 + 9k - 1 = 0$$

$$\Rightarrow (k+1)(10k-1) = 0$$

$$\Rightarrow k = -1 \, (or) k = \frac{1}{10}$$

(ii) $P\,[X \geq 6] = P[X = 6] + P[X = 7]$

$$= 2k^2 + 7k^2 + k$$

$$= 9k^2 + k$$

$$= \frac{9}{100} + \frac{1}{10}$$

$$= \frac{19}{100}$$

(iii) $P[X < 6] = 1 - P[X \geq 6]$

$$= 1 - \frac{19}{100}$$

$$= \frac{81}{100}$$

(iv) $P[1 < X < 5] = P[X = 2] + P[X = 3] + P[X = 4]$

$$= 2k + 2k + 3k$$

$$= 7k$$

$$= \frac{7}{10}$$

(v) $P[1.5 < X < 4.5 \,/\, X > 2] = \frac{P[1\cdot5 < X<4\cdot5 \cap X > 2]}{P[X>2]}$

$$= \frac{P[2 < X<4\cdot5\,]}{P[X>2]}$$

$$= \frac{P[X=3]+P[X=4]}{P[X>2]}$$

$$= \frac{\frac{5}{10}}{\frac{7}{10}}$$

$$= \frac{5}{7}$$

Distribution of X:

X	**P(x)**	**F(X) = P[X≤ x]**
0	0	F(0) = P[X≤ 0] = 0
1	k	F(1) = P[X≤ 1] = $F(0) + P(1) = 0 + \frac{1}{10} = \frac{1}{10}$
2	2k	F(2) = P[X≤ 2] = $F(1) + P(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$
3	2k	F(3) = P[X≤ 3] = $F(2) + P(3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10}$
4	3k	F(4) = P[X≤ 4] = $F(3) + P(4) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10}$

5	k^2	F(5) = P[X$\leq$ 5] = $F(4) + P(5) = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$
6	$2k^2$	F(6) = P[X$\leq$ 6] = $F(5) + P(6) = \frac{81}{100} + \frac{2}{100} = \frac{83}{100}$
7	$7k^2 + k$	F(7) = P[X$\leq$ 7] = $F(6) + P(7) = \frac{83}{100} + \frac{7}{100} + \frac{1}{10} = \frac{100}{100}$

The value of k = 4 when $P[X < k] > \frac{1}{2}$

3. If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$. Find the probability distribution.

Solution:

Let $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4) = k$

$\Rightarrow 2P(X = 1) = k$

$\Rightarrow P(X = 1) = \frac{k}{2}$

$\Rightarrow 3P(X = 2) = k$

$\Rightarrow P(X = 2) = \frac{k}{3}$

$\Rightarrow P(X = 3) = k$

$\Rightarrow 5P(X = 3) = k$

$\Rightarrow P(X = 3) = \frac{k}{5}$

We know that $\sum P(x) = 1$

$$\Rightarrow P(1) + P(2) + P(3) + P(4) = 1$$

$$\Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\Rightarrow \frac{15k+10k+30k+6k}{30} = 1$$

$$\Rightarrow \frac{61k}{30} = 1 \ \Rightarrow k = \frac{30}{61}$$

The Probability Distribution is

X	1	2	3	4
P(x)	$\frac{k}{2} = \frac{1}{2} \times \frac{30}{61} = \frac{15}{61}$	$\frac{k}{3} = \frac{1}{3} \times \frac{30}{61} = \frac{10}{61}$	$k = \frac{30}{61}$	$\frac{k}{5} = \frac{1}{5} \times \frac{30}{61} = \frac{6}{61}$

4. Suppose that the random variable X assumes three values 0, 1 and 2 with probabilities 1/3, 1/6 and ½ respectively. Obtain the distribution function of X.

Solution:

Values of X = x	0	1	2
P(x)	1/3	1/6	1/2
	P(0)	P(1)	P(2)

The distribution of X

X	**P(x)**	**F(X) = P[X$\leq x$]**
0	0	F(0) = P[X$\leq 0] = \frac{1}{3}$
1	k	F(1) = P[X$\leq 1] = F(0) + P(1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$
2	2k	F(2) = P[X$\leq 2] = F(1) + P(2) = \frac{1}{2} + \frac{1}{2} = 1$

5. The Probability function of an infinite distribution is given by $P(X = j) = \left(\frac{1}{2}\right)^j, j = 1, 2, \ldots\ldots, \infty$. Verify that the total probability is 1 and find also mean, variance, P(X is even), P(X is divisible by 3), P(X ≥ 5)

Solution:

x	1	2	3	4	5	….
P(X = x)	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$	…..

$$\sum p(x) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots\ldots$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots.\right] = \frac{1}{2}\left[\left(1 - \frac{1}{2}\right)^{-1}\right] = \frac{1/2}{1/2} = 1$$

$\therefore$ Total Probability is 1

Mean , $E(X) = \sum xp(x)$

$= 1 \times \frac{1}{2} + 2 \times \left(\frac{1}{2}\right)^2 + 3 \times \left(\frac{1}{2}\right)^3 + 4 \times \left(\frac{1}{2}\right)^4 + \cdots \ldots$

$= \frac{1}{2}\left[1 + 2 \times \frac{1}{2} + 3\left(\frac{1}{2}\right)^2 + \cdots .\right]$

$= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right)^{-2}\right]$ $(\because 1 + 2x + 3x^2 + \cdots . = (1-x)^{-2})$

$= \frac{1/2}{1/4} = 2$

$$\therefore E(X) = 2$$

$$E(X^2) = \sum x^2 p(x)$$

$= 1^2 \times \frac{1}{2} + 2^2 \times \left(\frac{1}{2}\right)^2 + 3^2 \times \left(\frac{1}{2}\right)^3 + 4^2 \times \left(\frac{1}{2}\right)^4 + \cdots \ldots$

$= \frac{1}{2}\left[1 + 4 \times \frac{1}{2} + 9\left(\frac{1}{2}\right)^2 + \cdots .\right]$

$= \frac{1}{2}\left[\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)^{-3}\right]$ $(\because 1 + 4x + 9x^2 + \cdots . = (1+x)(1-x)^{-3})$

$= \frac{1}{2} \times \frac{3}{2} \times \frac{8}{1} = 6$

$$Var(X) = E(X^2) - [E(X)]^2$$

$= 6 - 4 = 2$

P(X is even) = P(X= 2) + P(X = 4) + P(X = 6)+….

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots \ldots$$

$$= \left(\frac{1}{2}\right)^2 \left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots \ldots\right]$$

$$= \frac{1}{4}\left[\left(1 - \frac{1}{4}\right)^{-1}\right] = \frac{1}{4} \times \frac{4}{3}$$

P(X is even) = $\frac{1}{3}$

P(X is divisible by 3) = P(X=3) + P(X = 6) + P(X =9)+….

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \cdots \ldots$$

$$= \left(\frac{1}{2}\right)^3 \left[1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots \ldots\right]$$

$$= \frac{1}{8}\left[\left(1 - \frac{1}{8}\right)^{-1}\right] = \frac{1}{8} \times \frac{8}{7}$$

P(X is divisible by 3) = $\frac{1}{7}$

P(X $\geq$ 5) = P(X=5) + P(X = 6) + P(X =7)+….

$$= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^7 + \cdots \ldots$$

$$= \left(\frac{1}{2}\right)^5 \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots \ldots\right]$$

$$= \frac{1}{32}\left[\left(1 - \frac{1}{2}\right)^{-1}\right] = \frac{1}{32} \times 2 = \frac{1}{16}$$

MATHEMATICAL EXPECTATION FOR DISCRETE RANDOM VARIABLE

Note:

(i) $E(c) = c$

(ii) $Var(c) = 0$

(iii) $E(aX) = aE(X)$

(iv) $E(aX + b) = aE(X) + b$

(v) $Var(aX) = a^2Var(X)$

(vi) $Var(aX \pm b) = a^2Var(X)$

Problems:

1. **If $Var(X) = 4$, find $Var(4X + 5)$, where X is a random variable.**

Solution:

We know that $Var(aX + b) = a^2Var(X)$

Here $a = 4, Var(X) = 4$

$Var(4X + 5) = 4^2Var(X) = 16 \times 4 = 64$

2. **Let X be the number on a die when a die is thrown. Find the mean and variance of X.**

Solution:

The PMF is given by

x	1	2	3	4	5	6
P(X = x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Mean , $E(X) = \sum xp(x)$

$$= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$

$$= \frac{21}{6} = 3.5$$

$$E(X^2) = \sum x^2 p(x)$$

$$=1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6}$$

$$= \frac{91}{6} = 15.16$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 15.16 - (3.5)^2 = 2.91$$

3. A Fair coin is tossed three times. Let X be the number if tails appearing, Find the probability distribution of X. And also calculate E(X)

Solution:

S = { HHH, HTH,HHT,THH,TTH,THT,HTT,TTT}

X denotes the number of tail

∴ X(HHH) = 0,X(HTH) = 1, X(HHT) = 1, X(THH) = 1, X(TTH) = 2, X(HTT) = 2, X(TTT) = 3

∴ Random variable X takes the value 0,1,2,3

$$P(X = 0) = P(HHH) = \frac{1}{8}$$

$$P(X = 1) = P(HTH, HHT, THH) = \frac{3}{8}$$

$$P(X = 2) = P(TTH, THT, TTH) = \frac{3}{8}$$

$$P(X = 3) = P(TTT) = \frac{1}{8}$$

The probability function of X is

x	0	1	2	3
P(X = x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean , $E(X) = \sum xp(x)$

$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$

$= \frac{12}{8} = 1.5$

Continuous Random Variable:

If X is a random variable which can take all the values in an interval then X is called continuous random variable.

Properties of Probability Density Function:

The Probability density function of the random variable X denoted by f(x) has the following properties.

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

Cumulative Distribution Function (CDF):

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Properties of CDF:

(i) $F(-\infty) = 0$

(ii) $F(\infty) = 1$

(iii) $\frac{d}{dx}[F(x)] = f(x)$

(iv) $P(X \leq a) = F(a)$

(v) $P(X > a) = 1 - F(a)$

(vi) $P(a \leq X \leq b) = F(b) - F(a)$

Problems on Continuous Random Variables:

1. **A continuous random variable X has a density function $f(x) = \frac{K}{1+x^2}$, $-\infty \leq X \leq \infty$. i) Find the values of K ii) Cumulative distribution function iii) P(X > 0) iv) Mean of X.**

Solution: Given X is a continuous RV defined in $(-\infty, \infty)$

i) To find of k

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{K}{1+x^2} dx = 1$$

$$\Rightarrow K \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$\Rightarrow K \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$\Rightarrow K[tan^{-1}x]_{-\infty}^{\infty} = 1$$

$$\Rightarrow K[tan^{-1}\infty - tan^{-1}(-\infty)] = 1$$

$$\Rightarrow K\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1$$

$$\Rightarrow K\left[\frac{2\pi}{2}\right] = 1$$

$$\Rightarrow K = \frac{1}{\pi}$$

ii) CDF of X

$$F(x) = P(X \leq x) = k \int_{-\infty}^{x} \frac{dx}{1+x^2}$$

$= \frac{1}{\pi}[tan^{-1}x]_{-\infty}^{x}$

$$= \frac{1}{\pi}(\, tan^{-1}x - tan^{-1}(-\infty))$$

$$F(x) = \frac{1}{\pi}\left(tan^{-1}x + \frac{\pi}{2}\right);\ -\infty < x < \infty$$

iii) $P(X > 0) = \int_0^{\infty} f(x)dx = k\int_0^{\infty} \frac{dx}{1+x^2}$

$= \frac{1}{\pi}[tan^{-1}x]_0^{\infty}$

$$= \frac{1}{\pi}(\, tan^{-1}\infty - tan^{-1}0) = \frac{1}{2}$$

$$iv)\ Mean\ of\ X, E(X) = \int_{-\infty}^{x} xf(x)dx$$

$$= \int_{-\infty}^{\infty} \frac{x\,dx}{1+x^2}$$

$E(X) = 0 \qquad \because \frac{x}{1+x^2}\ is\ an\ odd\ function$

2. **A Continuous random variable X can assume any value between X = 2 and X = 5 has the density function given by $\boldsymbol{f(x) = k(1+x). Find\ \ i)\ k\ \ ii)\ p[\,X > 4]\quad iii)\ P\ [\,3 > X > 4]}$**

Solution:

Solution: Given X is a continuous RV defined in (2,5)

i) To find of k

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_2^5 K(1+x)dx = 1$$

$$\Rightarrow K\int_2^5 (1+x)dx = 1$$

$$\Rightarrow K\left[x+\frac{x^2}{2}\right]_2^5 = 1$$

$$\Rightarrow K\left[5+\frac{25}{2}-2-2\right] = 1$$

$$\Rightarrow K\left[\frac{27}{2}\right] = 1$$

$$\Rightarrow K = \frac{2}{27}$$

$$\boldsymbol{ii)\ p[\,X > 4] = \int_4^5 f(x)dx}$$

$$= \boldsymbol{K}\int_4^5 (1+x)dx$$

$$= K\left[x+\frac{x^2}{2}\right]_4^5$$

$$\Rightarrow K\left[5+\frac{25}{2}-4-8\right]$$

$$\Rightarrow K\left[\frac{25}{2}-7\right]$$

$$=\frac{2}{27}\left[\frac{11}{2}\right]=\frac{11}{27}$$

$$\boldsymbol{iii)}\ \boldsymbol{P}\ [\,\mathbf{3}>\boldsymbol{X}>\mathbf{4}]=\int_{3}^{4} f(x)dx$$

$$=\boldsymbol{K}\int_{3}^{4}(1+x)dx$$

$$=K\left[x+\frac{x^2}{2}\right]_3^4$$

$$=K\left[4+8-3-\frac{9}{2}\right]$$

$$=\frac{2}{27}\left[\frac{9}{2}\right]=\frac{1}{3}$$

$$\boldsymbol{P}\ [\,\mathbf{3}>\boldsymbol{X}>\mathbf{4}]=\frac{1}{3}$$

3. If a random variable X has PDF $\boldsymbol{f(x)}=\begin{cases}\frac{\mathbf{1}}{\mathbf{4}}, |\boldsymbol{x}|<\mathbf{2}\\ \mathbf{0}, |\boldsymbol{x}|>\mathbf{2}\end{cases}$

Find (i) $\boldsymbol{P[X<1]}$ $\boldsymbol{ii)P[|X|>1]}$ $\boldsymbol{(iii)P[2X+3>5]}$

Solution:

(i) $P[X<1]=\int_{-2}^{1} f(x)dx$

$$= \int_{-2}^{1} \frac{1}{4} dx$$

$$= \frac{1}{4} [x]_{-2}^{1}$$

$$= \frac{1}{4} [1 - (-2)]$$

$$= \frac{3}{4}$$

(ii) $P[|X| > 1] = 1 - P[-1 < X < 1]$

$$= 1 - \int_{-1}^{1} f(x) dx$$

$$= 1 - \int_{-1}^{1} \frac{1}{4} dx$$

$$= 1 - \frac{1}{4} [x]_{-1}^{1}$$

$$= 1 - \frac{1}{4} [1 - (-1)]$$

$$= 1 - \frac{2}{4}$$

$$= \frac{2}{4}$$

(iii) $P[2X + 3 > 5] = P[2X > 5 - 3]$

$$= P\left[X > \frac{5-3}{2}\right]$$

$$= P\left[X > \frac{2}{2}\right]$$

$$= P[X > 1]$$

$$= \int_{1}^{2} f(x) dx$$

$$= \int_{1}^{2} \frac{1}{4} dx$$

$$= \frac{1}{4} [x]_{1}^{2}$$

$$= \frac{1}{4}[2-(1)] \quad = \frac{1}{4}$$

MATHEMATICAL EXPECTATION OF CONTINUOUS RANDOM VARIABLES

(i) $E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$

(ii) $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x) dx$

(iii) $Var(X) = E(X^2) - E[(X)]^2$

Problems:

1. Let X be a continuous random variable with probability density function $\boldsymbol{f(x) = kx(2-x), 0 < x < 2}$. Find (i) k (ii) mean (iii) variance (iv) cumulative distribution function of X (v) rth moment.

Solution:

(i) To find k,

$$\int_0^2 f(x) dx = 1 \Rightarrow k \int_0^2 (2x - x^2) dx = 1$$

$$\Rightarrow k \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow k \left[4 - \frac{8}{3}\right] = 1$$

$$\Rightarrow k \left(\frac{4}{3}\right) = 1$$

$$\Rightarrow k = \frac{3}{4}$$

(ii) To calculate mean of X

$$E(X) = \int_0^2 x\, f(x)dx$$

$$= \int_0^2 x^2\ \frac{3}{4}(2 - x)dx$$

$$= \frac{3}{4}\int_0^2 (2x^2 - x^3)dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left(\frac{16}{3} - 4\right)$$

$$= \frac{3}{4} \times \frac{4}{3} = 1$$

(iii) To calculate variance of X

$$E(X^2) = \int_0^2 x^2\, f(x)dx$$

$$= \int_0^2 x^3\ \frac{3}{4}(2 - x)dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)dx$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left(8 - \frac{32}{5}\right)$$

$$= \frac{3}{4} \times \frac{8}{5} = \frac{6}{5}$$

$$Var(X) = E(X^2) - E[(X)]^2$$

$$= \frac{6}{5} - 1 = \frac{1}{5}$$

(iv) To calculate CDF of X

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_0^x f(x)dx$$

$$= \int_0^x \frac{3}{4} x(2-x)dx$$

$$= \frac{3}{4} \int_0^x (2x - x^2)dx$$

$$= \frac{3}{4} \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^x$$

$$= \frac{3}{4}\left(x^2 - \frac{x^3}{2}\right)$$

$$= \frac{1}{4}(3x^2 - x^3)$$

$$F(x) = \begin{cases} 0; & x < 0 \\ \frac{1}{4}(3x^2 - x^3); & 0 \le x < 2 \\ 1; & x \ge 2 \end{cases}$$

(v) To find the rth moment:

$$E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$

$$= \int_0^2 x^r \ \frac{3}{4} x(2-x)dx$$

$$= \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\, dx$$

$$= \frac{3}{4}\left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(2\,\frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right) - (0-0)\right]$$

$$= \frac{3}{4} \times 2^r 2^2 \left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= 6 \cdot \frac{2^r}{(r+2)(r+3)}$$

2. The probability distribution function of a random variable X is

$$\boldsymbol{f(x) = \begin{cases} x; & 0 < x < 1 \\ 2 - x; & 1 < x < 2 \\ 0; & x > 2 \end{cases}}$$ **Find the cumulative distribution function of X.**

Solution:

We know that c.d.f $F(x) = \int_{-\infty}^{x} f(x)dx$

(i) When $\boldsymbol{0 < x < 1}$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_0^x f(x)dx$$

$$= 0 + \int_0^x x\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{x^2}{2} - 0$$

$$= \frac{x^2}{2}$$

(ii) When $1 < x < 2$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x)dx$$

$$= 0 + \int_{0}^{1} f(x)dx + \int_{1}^{x} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \left(\frac{1}{2} - 0\right) + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

(iii) When $x > 2$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{x} f(x)dx$$

$$= 0 + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{x} x\,dx$$

$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{2} (2 - x)dx + 0$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2 - \frac{3}{2} \quad = 1$$

$$F(x) = \begin{cases} \frac{x^2}{2}; & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1; & 1 < x < 2 \\ 1; & x > 2 \end{cases}$$

3. The Cumulative distribution function of a random variable X is given by

$$\boldsymbol{F(x) = \begin{cases} 0; & x < 0 \\ x^2; & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2; & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}}$$

Find the pdf of X and evaluate $P(|X| \le 1)$ using both pdf and cdf.

Solution:

Given

$$F(x) = \begin{cases} 0; & x < 0 \\ x^2; & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2; & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

Pdf id $f(x) = \frac{d}{dx}[F(x)]$

$$f(x) = \begin{cases} 0; & x < 0 \\ 2x; & 0 \le x < \frac{1}{2} \\ \frac{6}{25}(3-x); & \frac{1}{2} \le x < 3 \\ 0, & x \ge 3 \end{cases}$$

To find $P(|X| \le 1)$ using cdf:

$$P(|X| \le 1) = P(-1 \le X \le 1)$$

$$= F(1) - F(-1)$$

$$= \left[1 - \frac{3}{25}(3-1)^2\right] - 0$$

$$= 1 - \frac{12}{25}$$

$$= \frac{25-12}{25} = \frac{13}{25}$$

To find $P(|X| \le 1)$ using pdf:

$$P(|X| \le 1) = P(-1 \le X \le 1)$$

$$= \int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{0} f(x)dx + \int_{0}^{\frac{1}{2}} f(x)dx + \int_{\frac{1}{2}}^{1} f(x)dx$$

$$= 0 + \int_{0}^{\frac{1}{2}} 2x\,dx + \int_{\frac{1}{2}}^{1} \frac{6}{25}(3-x)dx$$

$$= 2\left(\frac{x^2}{2}\right)_0^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{4} + \frac{6}{25}\left[3 - \frac{1}{2} - \frac{3}{2} + \frac{1}{8}\right]$$

$$= \frac{13}{25}$$

1.2 Baye's Theorem

If $B_1, B_2, \ldots, B_n$ be a set of exhaustive and mutually exclusive events associated with a random experiment and D is another event associated with B_i, then

$P(D/B_i) = \frac{P(B_i) \cdot P(D/B_i)}{\sum_{i=1}^{n} P(B_i)\, P(D/B_i)}$

State and Prove Bayes Theorem

(OR)

State and Prove Theorem of Probability of Causes.

Soln :

Statement

If $B_1, B_2, \ldots$ B_n be a set of exhaustive and mutually exclusive events associated with random experiment and D is another event associated with (or caused)by Bi. Then

$$P(D \mid B_i) = \frac{P(B_i) \cdot P(D/B_i)}{\sum_{i=1}^{n} P(B_i) P(D/B_i)}$$

Proof :

$$P(B_i \cap D) = P(B_i) \cdot P\left(\frac{D}{B_i}\right)$$

$$P(D \cap B_i) = P(D) \cdot P\left(\frac{B_i}{D}\right)$$

$$P(B_i \mid D) = \frac{P(B_i) \cdot P(D/B_i)}{P(D)} \ldots \ldots \ldots (1)$$

The inner circle reprosents the events D.D can occur along with $B_1, B_2, \ldots B_n$ that are exhaustive and mutually exclusive

$\therefore DB_1, DB_1, \ldots . DB_n$ are also mutually exclusive such that

$D = DB_1 + DB_2 + \cdots + DB_n$

$\therefore D = \sum DB_i$

$P[D] = P[\Sigma DB_i]$

$= \sum P[DB_i]$

$= \sum P[D \cap B_\mathrm{i}]$

$P[D] = \sum_{i=1}^{n} P(B_i) \cdot P(D/B_i)$

Substitute $P[D]$ in eqn (1)

$$(1) \Rightarrow P[B_i/D] = \frac{P(B_i) \cdot P(D/B_i)}{\sum_{i=1}^{n} P(B_\mathrm{i}) P(D/B_i)}$$

Hence the proof ,

Problem based on Baye's Theorem

1. **Four boxes A,B,C,D contain fuses. The boxes contain 5000, 3000, 2000 and 1000 fuses respectively. The percentages of fuses in boxes which are defective are 3%, 2%, 1% and 5% respectively.one fuse in selected at random arbitrarily from one of the boxes. It is found to be defective fuse. Find the probability that it has come from box D.**

(OR)

Four boxes A,B,C,D contain fuses. Box A contain 5000 fuses , box B contain 3000 fuses, box C contain 2000 fuses and box D contain 1000 fuses. The percentage of fuses in boxes which are defective are 3%, 2%, 1% and 0.5% respectively. One fuse is select at random from one of the boxes. It is found to be defective fuse . What is the probability that it has come from box D.

Soln:

Since selection ratio is not given

Assume selection ratio is 1 : 1 : 1 : 1

Total = 1+1+1+1 = 4

$P(A) = \frac{1}{4}$

$P(B) = \frac{1}{4}$

$P(C) = \frac{1}{4}$

$P(D) = 1/4$

Let E be the event selecting a defective fuse from any one of the machine

$P(E/A) = 3\% = 0.03$

$P(E/B) = 2\% = 0.02$

$P(E/C) = 1\% = 0.01$

$P(E/D) = 5\% = 0.05$

$$P(E) = P(A)P(E/A) + P(B)P(E/B) + P(C)P(F/\mathrm{C}) + P(D)(F/\mathrm{D})$$

$$= \frac{1}{4} \times 0.03 + 1/4 \times 0.02 + 1/4 \times 0.01 + 1/4 \times 0.05$$

$$= 0.0275$$

$$P(D/E) = \frac{P(D)P(E/D)}{P(E)}$$

$$= \frac{\frac{1}{4}\times 0.05}{0.0275} = 0.4545$$

$$= 0.4545$$

2. **In a bolt Factory, Machines A,B and C manufacture respectively 25%, 35% and 40% of total output . also out of these output of A,B,C are 5,4,2 percent respectively are defective. A bolt is drawn at random from the total output and it is found to be defective. What is the probability that it was manufactured by the machine B?**

(OR)

In a company machine A, B and C manufactured bolts, 25%, 35% and 40% of total output . also out of these output of A,B,C are 5,4,2 percent respectively are defective. A bolt is taken random from the total output and

it is found to be defective. Find the probability that it was manufactured by the machine B?

Soln:

Given ,P(E_1) =P(A) =25% = 0.25

P(E_2) =P(B) =35% = 0.35

P(E_3) =P(C) =40% = 0.40

Let D be the event of drawing defective bolt

$P(D/E_1) = 5\% = \frac{5}{100} = 0.05$

$P(D/E_2) = 4\% = 0.04$

$P(D/E_3) = 2\%) = 0 \cdot 02$

To find $P(\ E_2/D)$

By Bayes theorem

$$P(\ E_2/D) = \frac{P(E_2)P(D/E_2)}{P(E_1)\ P(D/E_1) + P(E_2)P(D/E_2) + P(E_3)P(D/E_3)}$$

$= \frac{(0.35)(0.04)}{(0.25)(0.05)+(0.35)(0.04)+(0.4)(0.02)}$

$= \frac{0.014}{0.0345}$

= 0.406

3. **A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the**

bag and is found to be red. Find the Probability that it was drawn from bag B

(OR)

A box A contains 2 white and 3 red balls and a box B contains 4 white and 5 red balls at random one ball is taking and is found to be red. What is the probability that it was drawn from bag B?

Soln:

Let B_1 be the event that the ball is drawn from the bag A.

Let B_2 be the event that the ball is drawn from the bag B.

Lat A be the event that the drawn ball is red

$P(B_1) = P(B_2) = \frac{1}{2}$

$P(A/B_1) = \frac{3C_1}{5C_1} = \frac{3}{5}$

$P(A/B_2) = \frac{5C_1}{9C_1} = \frac{5}{9}$

$$P(B_2/A) = \frac{P(B_2)P(A/B_2)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2)}$$

$$= \frac{(\frac{1}{2})(\frac{5}{9})}{\left(\frac{1}{2}\right)\left(\frac{3}{5}\right)+(\frac{1}{2})(\frac{5}{9})}$$

$$= \frac{\frac{5}{18}}{\frac{52}{90}}$$

$P(B_2/A)= \frac{25}{52}$