Electric flux density:

As stated earlier electric field intensity or simply ‘Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$$\vec{D} = \varepsilon\vec{E} \quad \text{.....................................(2.11)}$$

We define the electric flux ⍰ as

$$\psi = \int_S \vec{D}.d\vec{s} \quad \text{.....................................(2.12)}$$

Gauss's Law: Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

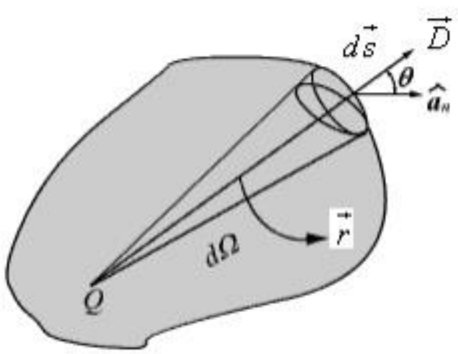

Fig 2.3: Gauss's Law

Let us consider a point charge Q located in an isotropic homogeneous medium of dielectric constant . The flux density at a distance r on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r \quad \text{.....................................(2.13)}$$

If we consider an elementary area ds, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2} ds\cos\theta \quad \text{.....................................(2.14)}$$

But $\frac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of *Q*. Therefore we can write $d\psi = \frac{Q}{4\pi} d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_S d\psi = \frac{Q}{4\pi}\oint_S d\Omega = Q$

which can seen to be same as what we have stated in the definition of Gauss's Law.

Application of Gauss's Law

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1.An infinite line charge

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density L2C/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 2.4(a) (next slide). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 2.4(b) (next slide).

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write, Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

Considering the fact that the unit normal vector to areas $S1$ and $S3$ are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we can write,

$$\rho_L l = \varepsilon_0 E . 2\pi\rho l$$

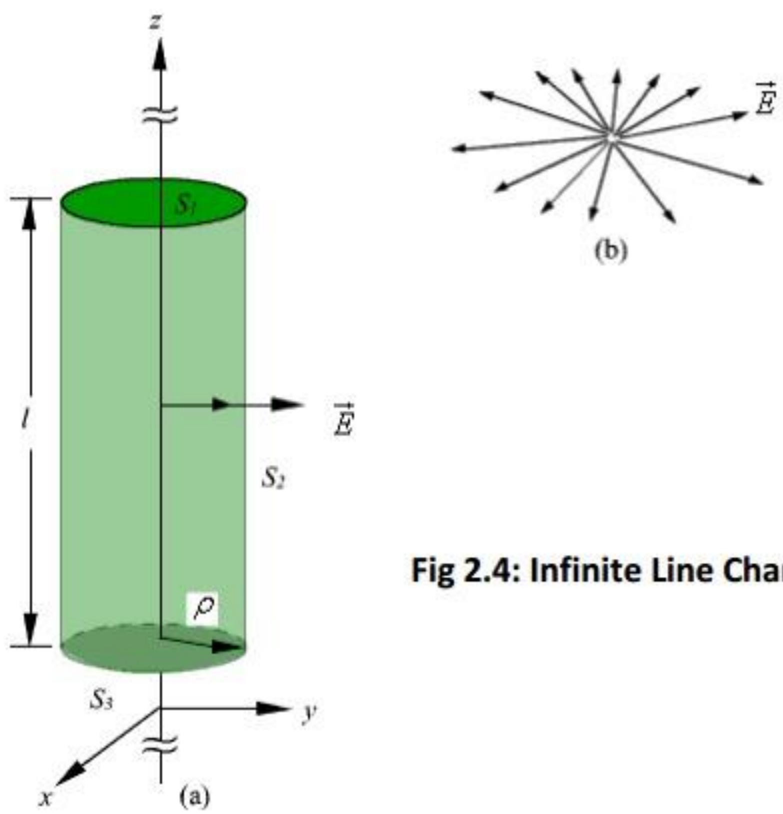

Fig 2.4: Infinite Line Charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho \quad \text{.....................................(2.16)}$$

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x*-*z* plane as shown in figure 2.5.

Assuming a surface charge density of □s for the infinite surface charge, if we consider a cylindrical volume having sides □s placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D}\cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y \qquad ....(2.17)$$

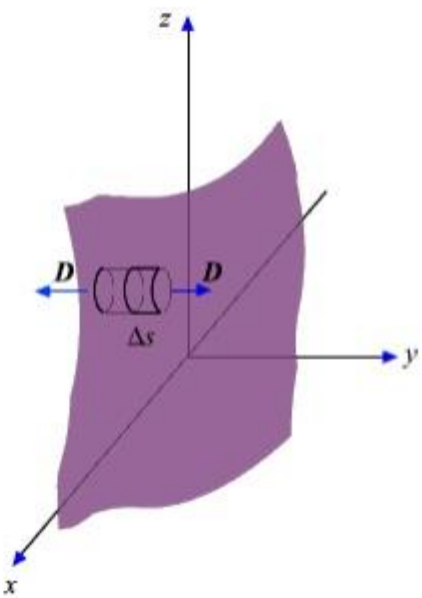

Fig 2.5: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

3. Uniformly Charged Sphere

Let us consider a sphere of radius r_0 having a uniform volume charge density of ρ_v C/m^3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius $r < r_0$ and $r > r_0$ as shown in Fig. 2.6 (a) and Fig. 2.6(b).

For the region $r \leq r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3 \quad \text{..........................(2.18)}$$

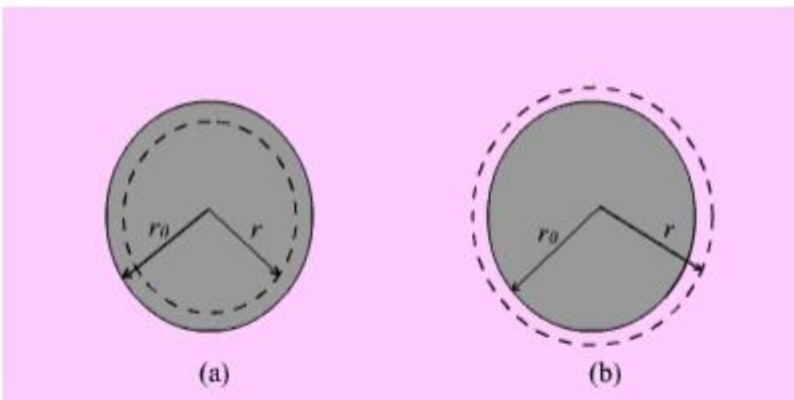

Fig 2.6: Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en} \quad \text{...............(2.19)}$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0 \quad \text{...............(2.20)}$$

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3 \quad \text{...............(2.21)}$$

By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0 \quad \text{...............(2.22)}$$

Electrostatic Potential and Equipotential Surfaces

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field.

Let us suppose that we wish to move a positive test charge c2 from a point *P* to another point *Q* as shown in the Fig. 2.8.

The force at any point along its path would cause the particle to accelerate and move it out of the region if unconstrained. Since we are dealing with an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while □c2 moves from *P* to *Q*. The work done by this external agent in moving the charge by a distance Bar dl is given by:

$$dW = -\Delta q \vec{E}.d\vec{l} \text{ ....(2.23)}$$

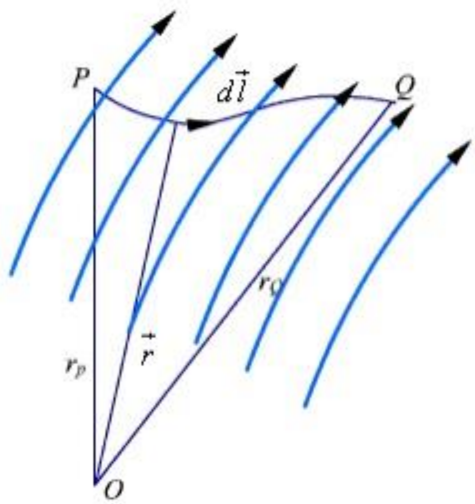

Fig 2.8: Movement of Test Charge in Electric Field

The negative sign accounts for the fact that work is done on the system by the external agent.

$$W = -\Delta q \int_P^Q \vec{E} \cdot d\vec{l} \text{ ....(2.24)}$$

The potential difference between two points *P* and *Q* , *VPQ*, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_P^Q \vec{E} \cdot d\vec{l} \text{ ....(2.25)}$$

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in Joules/Coulomb which is referred to as **Volts**.

Let us consider a point charge Q as shown in the Fig. 2.9.

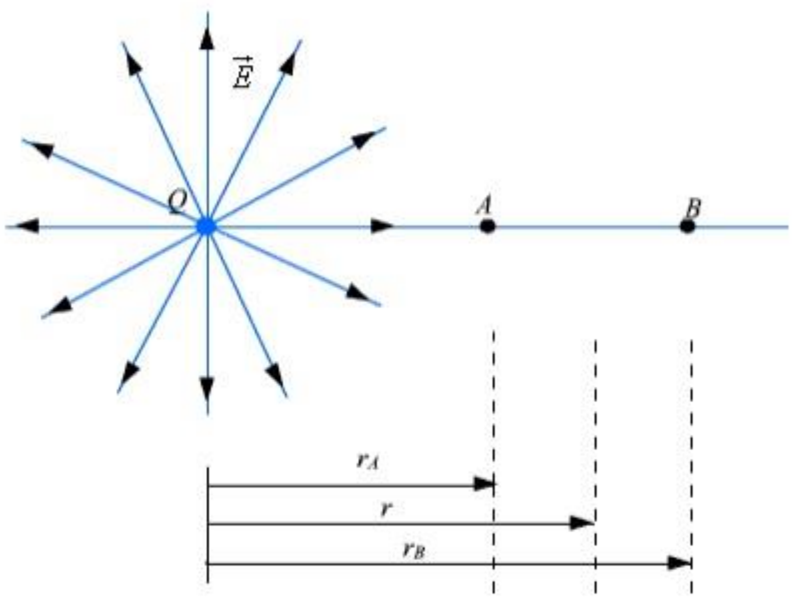

Fig 2.9: Electrostatic Potential calculation for a point charge

Further consider the two points A and B as shown in the Fig. 2.9. Considering the movement of a unit positive test charge from B to A , we can write an expression for the potential difference as:

$$V_{BA} = -\int_B^A \vec{E} \cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B$$

..................................(2.26)

It is customary to choose the potential to be zero at infinity. Thus potential at any point ($rA = r$) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. $rB = 0$).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \quad \text{.................................(2.27)}$$

Or, in other words,

$$V = -\int_{\infty}^{r} E.dl \quad \text{.................................(2.28)}$$

Let us now consider a situation where the point charge *Q* is not located at the origin as shown in Fig. 2.10.

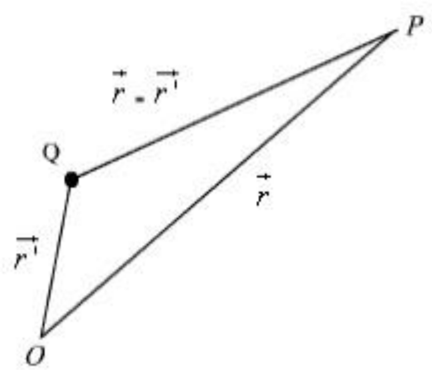

Fig 2.10: Electrostatic Potential due a Displaced Charge

The potential at a point *P* becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\left|\vec{r} - \vec{r}'\right|} \quad \text{.................................(2.29)}$$

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distribution also. Let us first consider *N* point charges *Q1, Q2,.....QN* located at points with position vectors $\vec{r_1}, \vec{r_2}, ...\vec{r_N}$. The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\left|\vec{r}-\vec{r_1}\right|} + \frac{Q_2}{\left|\vec{r}-\vec{r_2}\right|} + \cdots \frac{Q_N}{\left|\vec{r}-\vec{r_N}\right|}\right) \qquad \text{.................................(2.30a)}$$

or,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{\left|\vec{r}-\vec{r_n}\right|} \qquad \text{...(2.30b)}$$

For continuous charge distribution, we replace point charges Q_n by corresponding charge elements $\rho_L dl$ or $\rho_S ds$ or $\rho_V dv$

depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

For line charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}\,')dl'}{\left|\vec{r}-\vec{r_n}\right|} \qquad \text{..................................(2.31)}$$

For surface charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\vec{r}\,')ds'}{\left|\vec{r}-\vec{r_n}\right|} \qquad \text{.................................(2.32)}$$

For volume charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\vec{r}\,')dv'}{\left|\vec{r}-\vec{r_n}\right|} \qquad \text{.................................(2.33)}$$

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential at infinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C \quad \text{.................................(2.34)}$$

where C is a constant. In the same manner when potential is computed from a known electric field we can write:

$$V = -\int \vec{E} \cdot d\vec{l} + C \quad \text{.................................(2.35)}$$

The potential difference is however independent of the choice of reference.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l} = \frac{W}{Q} \quad \text{.......................(2.36)}$$

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point $P1$ to $P2$ in one path and then from point $P2$ back to $P1$ over a different path. If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position $P1$. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable, the work must have to be independent of path and depends on the initial and final positions.

Since the potential difference is independent of the paths taken, $VAB = -VBA$, and over a closed path,

$V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{l} = 0$ ....(2.37)

Applying Stokes's theorem, we can write:

$\oint \vec{E} \cdot d\vec{l} = \int_s \left(\nabla \times \vec{E} \right) \cdot d\vec{s} = 0$ ....(2.38)

from which it follows that for electrostatic field,

$\nabla \times \vec{E} = 0$ ....(2.39)

Any vector field $\vec{A}$ that satisfies $\nabla \times \vec{A} = 0$ is called an irrotational field.

From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial x} dz = -\vec{E} \cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z \right) \cdot \left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z \right) = -\vec{E} \cdot d\vec{l}$$

$\nabla V \cdot d\vec{l} = -\vec{E} \cdot d\vec{l}$ ....(2.40) from which we obtain,

$\vec{E} = -\nabla V$ ....(2.41)

From the foregoing discussions we observe that the electric field strength at any point is the negative of the potential gradient at any point, negative sign shows that $\vec{E}$ is directed from higher to lower values of $\vec{V}$. This gives us another method of computing the electric field, i. e. if we know the potential function, the electric field may be computed. We may note here that that one scalar function $\vec{V}$ contain all the information that three components of $\vec{E}$ carry, the same is possible because of the fact that three components of $\vec{E}$ are interrelated by the relation $\nabla \times \vec{E}$.

Example: Electric Dipole

An electric dipole consists of two point charges of equal magnitude but of opposite sign and separated by a small distance.

As shown in figure 2.11, the dipole is formed by the two point charges Q and -Q separated by a distance d , the charges being placed symmetrically about the origin. Let us consider a point P at a distance r, where we are interested to find the field.

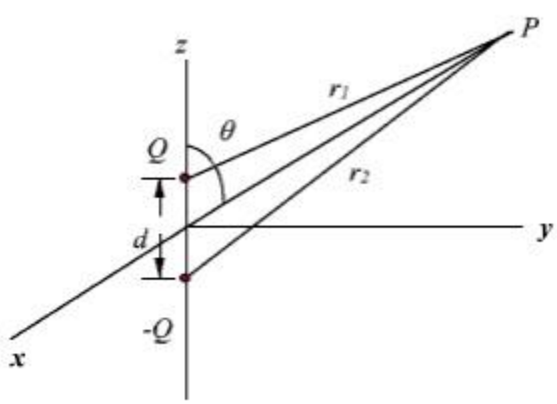

Fig 2.11 : Electric Dipole

The potential at P due to the dipole can be written as:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{Q}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \text{..........................(2.42)}$$

When r_1 and $r_2 >> d$, we can write $r_2 - r_1 = 2\times\frac{d}{2}\cos\theta = d\cos\theta$ and $r_1 \cong r_2 \cong r$.

Therefore,

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \text{..(2.43)}$$

We can write,

$$Qd\cos\theta = Qd\hat{a}_z \cdot \hat{a}_r \text{..(2.44)}$$

The quantity $\vec{P} = Q\vec{d}$ is called the **dipole moment** of the electric dipole.

Hence the expression for the electric potential can now be written as:

$$V = \frac{\vec{P}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2} \text{.................................(2.45)}$$

It may be noted that while potential of an isolated charge varies with distance as 1/*r* that of an electric dipole varies as 1/*r*2 with distance.

If the dipole is not centered at the origin, but the dipole center lies at $\vec{r'}$, the expression for the potential can be written as:

$$V = \frac{\vec{P}\cdot(\vec{r} - \vec{r'})}{4\pi\varepsilon_0\left|\vec{r} - \vec{r'}\right|^3} \text{.........................(2.46)}$$

The electric field for the dipole centered at the origin can be computed as

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta\right]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{a}_\theta$$

$$= \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$$\vec{E} = \frac{\vec{P}}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right) \qquad \text{........................(2.47)}$$

$$\vec{P} = Q\vec{d}$$

is the magnitude of the dipole moment. Once again we note that the electric field of electric dipole varies as $1/r3$ where as that of a point charge varies as $1/r2$.

Equipotential Surfaces

An equipotential surface refers to a surface where the potential is constant. The intersection of an equipotential surface with an plane surface results into a path called an equipotential line. No work is done in moving a charge from one point to the other along an equipotential line or surface.

In figure 2.12, the dashes lines show the equipotential lines for a positive point charge. By symmetry, the equipotential surfaces are spherical surfaces and the equipotential lines are circles. The solid lines show the flux lines or electric lines of force.

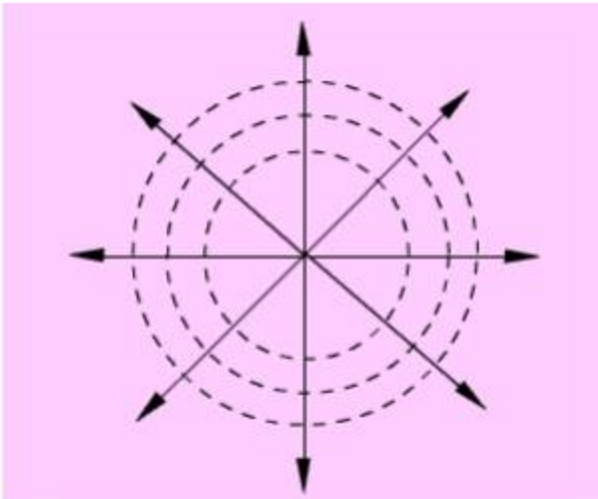

Fig 2.12: Equipotential Lines for a Positive Point Charge

Michael Faraday as a way of visualizing electric fields introduced flux lines. It may be seen that the electric flux lines and the equipotential lines are normal to each other.

In order to plot the equipotential lines for an electric dipole, we observe that for a given *Q* and *d*, a constant *V* requires that $\frac{\cos\theta}{r^2}$ is a constant. From this we can write $r = c_v\sqrt{\cos\theta}$ to be the equation for an equipotential surface and a family of surfaces can be generated for various values of c_v.When plotted in 2-D this would give equipotential lines.

To determine the equation for the electric field lines, we note that field lines represent the direction of $\vec{E}$ in space. Therefore,

$d\vec{l} = k\vec{E}$, k is a constant ....(2.48)

$\hat{a}_r dr + r d\theta \hat{a}_\theta + \hat{a}_\phi r \sin\theta = k(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) = d\vec{l}$ ....(2.49)

For the dipole under consideration $E_\phi = 0$, and therefore we can write,

$$\frac{dr}{E_r} = \frac{r d\theta}{E_\theta}$$

$$\frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta} = \frac{2d(\sin\theta)}{\sin\theta} \quad \text{...(2.50)}$$

Integrating the above expression we get $r = c_e \sin^2\theta$, which gives the equations for electric flux lines. The representative plot ($c_v = c$ assumed) of equipotential lines and flux lines for a dipole is shown in fig 2.13. Blue lines represent equipotential, red lines represent field lines.