HILBERT TRANSFORM

In Digital Signal Processing we often need to look at relationships between real and imaginary parts of a complex signal. These relationships are generally described by Hilbert transforms. Hilbert transform not only helps us relate the I and Q components but it is also used to create a special class of causal signals called analytic which are especially important in simulation. The analytic signals help us to represent band pass signals as complex signals which have specially attractive properties for signal processing. The role of Hilbert transform is to take the carrier which is a cosine wave and create a sine wave out of it. Now recall that the Fourier Series is written as

$$f(x) = \sum_{x=-\infty}^{x=+\infty} Cn \ e^{j\pi\omega t} \qquad (1)$$

Where $C_n = A_n + jB_n$ and $C_{-n} = A_n - jB_n$

Here An and Bn are the spectral amplitudes of cosine and sine waves. Now take a look at the phase spectrum. The phase spectrum is computed by

$$\Theta = \tan^{-1} (Bn/An) \qquad (2)$$

Cosine wave has no sine spectral content, so Bn is zero. The phase calculated is 90° for both positive and negative frequency from above formula. The wave has two spectral components each of magnitude A/2, both positive and lying in the real plane. Real plane is described as that passing vertically (R-V plane) and the Imaginary plane as one horizontally (R-I plane) through the Imaginary axis) Now compare Figure 1.4.1, in particular the spectral amplitudes. The cosine spectral amplitudes are both positive and lie in the real plane. The sine wave has spectral components that lie in the Imaginary plane and are of opposite sign. Now we wish to convert the cosine wave to a sine wave. There are two ways of doing that,

- ✓ Time domain
- ✓ Frequency domain.

To turn cosine into sine, we need to rotate the negative frequency component of the cosine by +90° and the positive frequency component by -90°. We will need to rotate the +Q phasor by -90° or in other words multiply it by -j. We also need to rotate the -Q phasor by +90° or multiply it by j. We can describe this transformation process called the Hilbert Transform as follows: All negative frequencies of a signal get a +90° phase shift and all positive frequencies get a -90° phase shift.

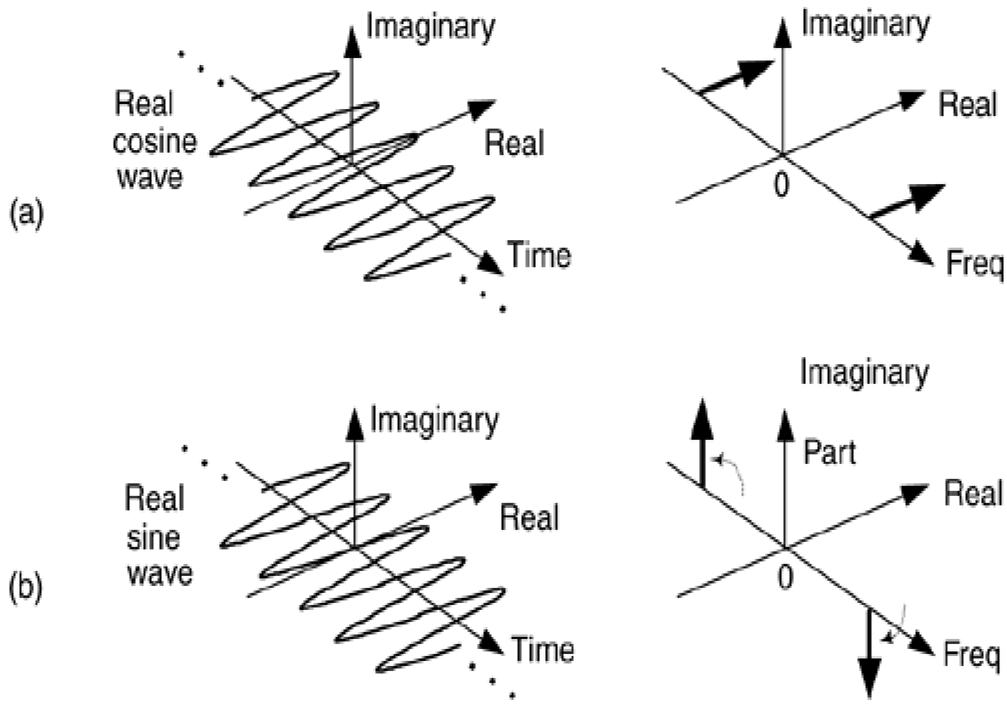

Figure 1.4.1 Hilbert Transform Cosine and Sine Spectral Amplitudes

Diagram Source Flylib.com

A negative cosine will come out a negative sine wave and one more transformation will return it to the original cosine wave, each time its phase being changed by 90°. For this reason Hilbert transform is also called a "quadrature filter".A peculiar sort of filter that changes the phase of the spectral components depending on the sign of their frequency. It only effects the phase of the signal. It has no effect on the amplitude at all.

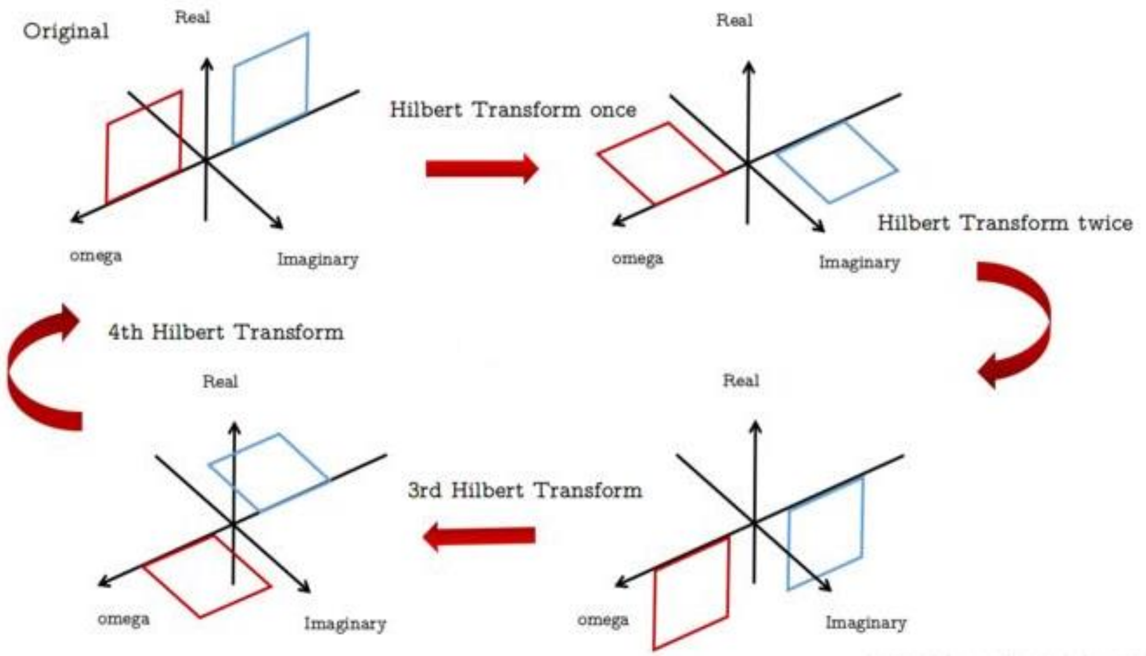

Figure 1.4.2 Hilbert Transform Process

Diagram Source Programmer sought

Hilbert transform of a signal x(t) is defined as the transform in which phase angle of all components of the signal is shifted by ±90o Hilbert transform of x(t) is represented with x^(t) shown in figure 1.4.2 it is given by,

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(s)}{t-s} ds, \tag{3}$$

where the integral is the Cauchy principal value integral. The reconstruction formula

$$x(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{x}(s)}{t-s} ds, \tag{4}$$

defines the Hilbert inverse transform. X(t), x^(t) is called a Hilbert transform pair.

Hilbert transformer:

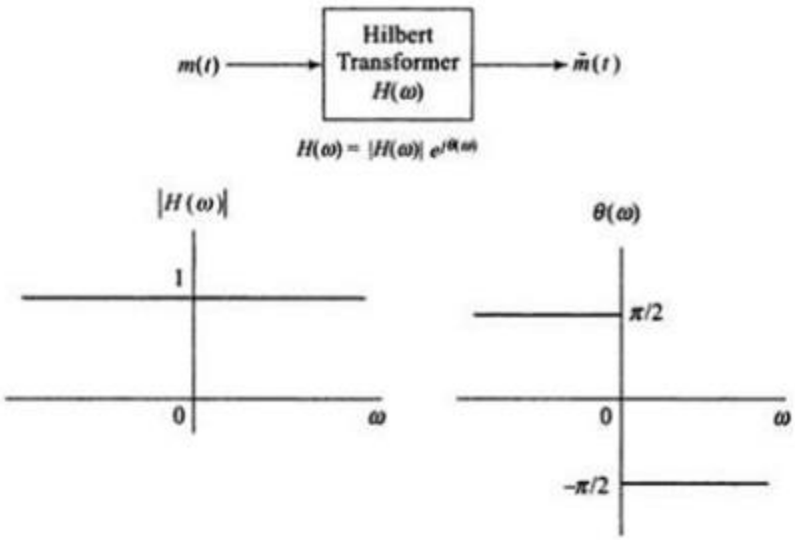

Figure 1.4.3 Magnitude and Phase Characteristics of Hilbert Transform

Diagram Source: Electronics Post

A Hilbert transformer produces a -90 degree phase shift for the positive frequency components of the input x(t), the amplitude doesn't change shown in figure 1.4.3.

Properties of the Hilbert transform:

A signal x(t) and its Hilbert transform $\hat{x}(t)$ have

1. the same amplitude spectrum
2. the same autocorrelation function
3. x(t) and $\hat{x}(t)$ are orthogonal
4. The Hilbert transform of $\hat{x}(t)$ is -x(t)

Pre envelope:

The pre envelope of a real signal x(t) is the complex function

$$x_+(t) = x(t) + j\ \hat{x}(t).$$

The pre envelope is useful in treating band pass signals and systems. This is due to the result

$$X_+(v) = \begin{cases} 2\,X(v), & v > 0 \\ X(0), & v = 0 \\ 0, & v < 0 \end{cases}$$

Complex envelope:

The complex envelope of a band pass signal x(t) is

$$\tilde{x}(t) = x_+(t)\, e^{-j2\pi f c t}.$$

Properties of the Hilbert Transform

A signal x(t) and its Hilbert transform x^(t) have

- The same amplitude spectrum.
- The same autocorrelation function.
- The energy spectral density is same for both x(t) and x^(t).
- x(t) and x^(t) are orthogonal.
- The Hilbert transform of x^(t) is -x(t)
- If Fourier transform exist then Hilbert transform also exists for energy and power signals.