# Lattice:

A Lattice is a partially ordered set(Poset)  $(L, \leq)$  in which for every pair of elements  $a, b \in L$ , both greatest lower bound (GLB) and least upper bound (LUB) exists.

## Note:

(i) GLB 
$$\{a, b\} = a * b$$
 (or)  $a \wedge b$  (or)  $a \cdot b$ 

(ii) LUB 
$$\{a, b\} = a \oplus b (or)a \lor b (or)a + b$$

**Properties of lattice:** 

Some important laws and its proof:

- (i) Idempotent law:
- $a \lor a = a, a \land a = a$
- (ii) Commutative law:

$$a \lor b = b \lor a$$
 and  $a \land b = b \land a$ PTIMIZE OUTSPRE

PALKULAM, KANYA

(iii) Associative law:

 $a \lor (b \lor c) = (a \lor b) \lor c$  and  $a \land (b \land c) = (a \land b) \land c$ 

# (iv) Absorption law:

 $a \lor (a \land b) = a \text{ and } a \land (a \lor b) = a$ 

MA8351 DISCRETE MATHEMATICS

## (v) Distributive law:

 $a \land (b \lor c) \ge (a \land b) \lor (a \land c)$ 

 $a \lor (b \land c) \le (a \lor b) \land (a \lor c)$ 

## Note:

i)  $a \leq a \lor b$  and  $b \leq a \lor b$ 

 $a \lor b$  is the upper bound of a and b.

If  $a \le c$  and  $b \le c$  then  $a \lor b \le c$ 

Hence  $a \lor b$  is the lub of a and b.

(ii)  $a \land b \leq a$  and  $a \land b \leq b$ .

 $a \wedge b$  is the lower bound of a and b.

If  $c \le a$  and  $c \le b$  then  $c \le a \land b^{4M}$ , KANYAKUMA

Hence  $a \wedge b$  is the glb of a and b.

ERVE OPTIMIZE OUTSPREND

INEERING

# Note:

If  $a \le b$  and  $a \le c$  then  $a \le b \lor c$ 

If  $a \leq b$  and  $a \leq c$  then  $a \leq b \wedge c$ 

## **Problems:**

MA8351 DISCRETE MATHEMATICS

## 1. State and prove Idempotent law:

Let  $(L, \land, \lor)$  be given lattice. Then, for any  $a, b, c \in L$ ,

 $a \lor a = a, a \land a = a$ .

**Proof:** 

EERING Given  $a \lor a = LUB(a, a) = LUB(a) = a$ 

Hence  $a \lor a = a$ 

Now,  $a \wedge a = \text{GLB}(a, a) = \text{GLB}(a) = a$ 

Hence  $a \wedge a = a$ 

Hence the proof.

2. State and prove Commutative law:

Let  $(L, \land, \lor)$  be given lattice. Then, for any  $a, b, c \in L$ ,

 $a \lor b = b \lor a$  and  $a \land b = b \land a$ ERVE OPTIMIZE OUTSPREAD

**Proof:** 

Given  $a \lor b = LUB(a, b) = LUB(b, a) = b \lor a$ 

Hence  $a \lor b = b \lor a$ 

Now,  $a \wedge b = \text{GLB}(a, b) = \text{GLB}(b, a) = b \wedge a$ 

Hence  $a \wedge b = b \wedge a$ 

Hence the proof.

# 3. State and prove Absorption law.

Prove that  $a \lor (a \land b) = a$  and  $a \land (a \lor b) = a$ 

(**or**)

ULAM, KANYAKUMA

**Proof:** 

We have  $a \land b \leq a$  and  $a \leq a$ 

 $\Rightarrow$  a is the upper bound of  $a \land b$  and a.

 $\Rightarrow a \lor (a \land b) \leq a \ldots (1)$ 

From the definition of lub we have

 $\Rightarrow a \leq a \lor (a \land b) \dots (2)$ 

From (1) and (2) we have  $a \lor (a \land b) = a$ OBSERVE OPTIMIZE OUTSPREND

Similarly we can prove that  $a \land (a \lor b) = a$ 

Hence the proof.

4. Every finite Lattice is bounded.

**Proof:** 

Let  $(L, \Lambda, \vee)$  be a given lattice.

Since L is a Lattice both GLB and LUB exist.

Let "a" be GLB of L and "b" be LUB of L.

Then for any  $x \in L$ , we have  $a \le x \le b$   $[= \dots (1)]$ 

From (1)

GLB  $\{a, x\} = a \land x = a$ 

LUB 
$$\{a, x\} = a \lor x = x$$

And

GLB 
$$\{x, b\} = x \land b =$$

LUB 
$$\{x, b\} = x \lor b = b$$

Therefore any finite lattice is bounded, KANYAKUN

X

Hence the proof.

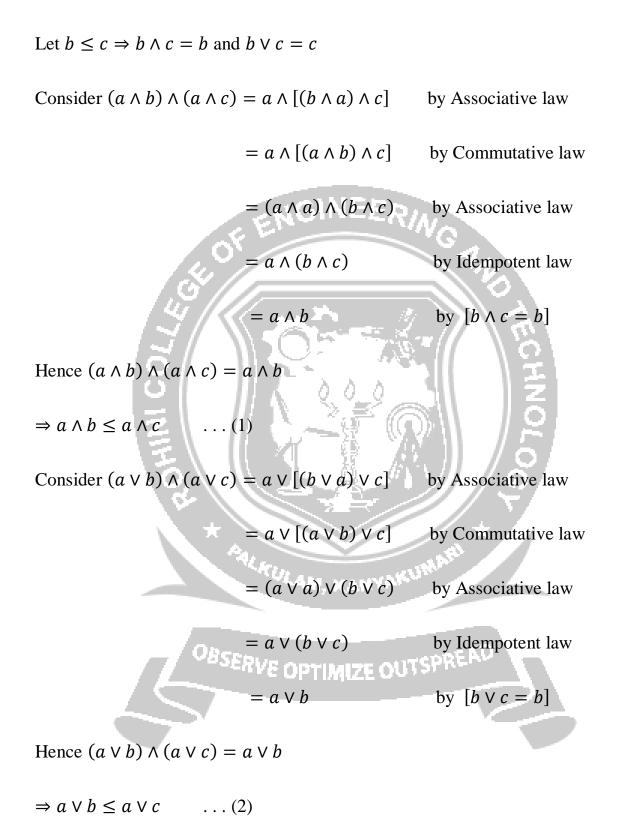
# OBSERVE OPTIMIZE OUTSPREAD

# 5. State and prove Isotonicity property.

Let  $(L, \leq)$  be a lattice. For any  $a, b, c \in L$  then  $b \leq c = \begin{cases} a \land b \leq a \land c \\ a \lor b \leq a \lor c \end{cases}$ 

# **Proof:**

By consistency law we have,  $a \le b \Leftrightarrow a \land b = a$  and  $a \lor b = a$ 



Hence the proof.

ERING

WLAM, KANYAKUN

## 6. State and prove Distributive law.

 $a \land (b \lor c) \ge (a \land b) \lor (a \land c)$ 

 $a \lor (b \land c) \le (a \lor b) \land (a \lor c)$ 

**Proof:** 

We know that  $a \wedge b \leq a$  and  $a \wedge b \leq b$ 

Also  $b \le b \lor c$ 

Hence  $a \land b \leq a$  and  $a \land b \leq b \leq b \lor c$ 

Hence  $a \wedge b$  is the lower bound of a and  $b \vee c$ 

 $\Rightarrow a \land b \leq a \land (b \lor c) \ldots (1)$ 

Again  $a \wedge c \leq a$  and  $a \wedge c \leq c$ 

Also  $c \le b \lor c$ 

Hence  $a \wedge c \leq a$  and  $a \wedge c \leq c \leq b \vee c$ BSERVE OPTIMIZE OUTSPREND

Hence  $a \wedge c$  is the lower bound of a and  $b \vee c$ .

 $\Rightarrow a \land c \leq a \land (b \lor c) \dots (2)$ 

From (1) and (2) we have

 $a \land (b \lor c)$  is the upper bound of  $a \land b$  and  $a \land c$ 

ERING

Hence  $(a \land b) \lor (a \land c) \le a \land (b \lor c)$ 

 $\Rightarrow a \land (b \lor c) \ge (a \land b) \lor (a \land c) \dots (I)$ 

We know that  $a \leq a \lor b$  and  $a \leq a \lor b$ 

Also  $b \wedge c \leq b$ 

Hence  $a \le a \lor b$  and  $b \land c \le b \le a \lor b$ 

Hence  $a \lor b$  is the lower bound of a and  $b \land c$ .

 $\Rightarrow a \lor (b \land c) \leq a \lor b \dots (3)$ 

Again  $a \le a \lor a$  and  $c \le a \lor c$ 

Also  $b \wedge c \leq c$ 

Hence  $a \le a \lor c$  and  $b \land c \le c \le a \lor c$ 

Hence  $a \lor c$  is the upper bound of a and  $b \land c$ .

 $\Rightarrow a \lor (b \land c) \leq a \lor \mathcal{G}_{BSERVE}^{(4)}$ 

From (3) and (4) we have

 $a \lor (b \land c)$  is the lower bound of  $a \lor b$  and  $a \lor c$ 

 $\Rightarrow a \lor (b \land c) \le (a \lor b) \land (a \lor c) \dots (II)$ 

Hence the proof.

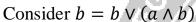
7. State and prove Cancellation law.

Let  $(L, \leq)$  be a distributive lattice. Then  $a \lor b = a \lor c$  and  $a \land b = a \land c \Rightarrow$ 

 $b = c \forall a, b, c \in L$ 

**Proof:** 

EERING By absorption law, we have  $a \lor (a \land b) = a$ 





OBSERVE OPTIMIZE OUTSPREAD

Hence the proof.

= c

8. State and prove Consistency Law.

Let  $(L, \leq)$  be a lattice. Then  $a \leq b \Leftrightarrow a \land b = a \Leftrightarrow a \lor b \forall a, b, c \in L$ 

**Proof:** 

ERING

HULAM, KANYAKUMA

## First we prove that $a \leq b \Leftrightarrow a \wedge b = a$

We assume that  $a \leq b$ 

To prove  $a \wedge b = a$ 

We have  $a \le b$  and  $a \le a$ 

 $\Rightarrow$  a is the lower bound of a and b.

 $\Rightarrow a \leq a \wedge b$ 

By the definition of greatest lower bound

...(1)

$$\Rightarrow a \land b \le a \qquad \dots (2)$$

From (1) and (2) we have,  $a = a \wedge b$ 

Conversely assume that  $a = a \wedge b$ 

To prove  $a \le b$ 

This is possible only when  $a \le b$ BSERVE OPTIMIZE OUTSPREAD

Hence  $a \le b \Leftrightarrow a \land b = a$ 

Next we prove that  $a \wedge b = a \Leftrightarrow a \vee b = b$ 

Assume that  $a \wedge b = a$ 

To prove  $a \lor b = b$ 

By absorption law  $a \lor (a \land b) = a$  and  $a \land (a \lor b) = a$ 

Consider  $b = b \lor (a \land b)$ 

 $= b \lor a$ 

Hence  $a \lor b = b$ 

Conversely assume that  $a \lor b = b$ 

To prove  $a \wedge b = a$ 

By absorption law  $a \land (a \lor b) = a$ 

Consider  $a = a \land (a \lor b)$ 

 $= a \wedge b$ 

Hence  $a \wedge b = a \Leftrightarrow a \vee b = b$ 

9. Show that a chain is a lattice. 4M, KANYAKUMP

**Proof:** 

OBSERVE OPTIMIZE OUTSPREAD

IEERING

Let  $(L, \leq)$  be a lattice.

If  $a, b \in L$  then  $a \leq b$  or  $b \leq a$ 

If  $a \le b$  then  $a \land b = a$  and  $a \lor b = b$ 

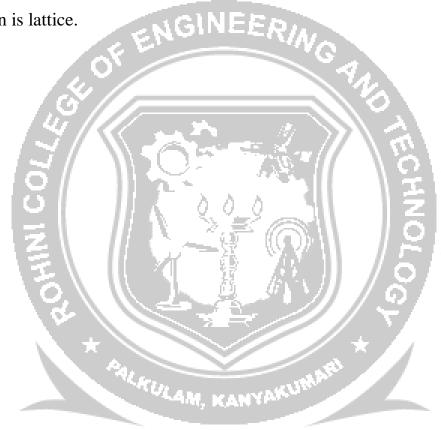
Therefore GLB and LUB of a and b exists.

If  $b \le a$  then  $b \land a = b$  and  $b \lor a = a$ 

Therefore GLB and LUB of a and b exists.

Hence every pair of elements has a GLB and LUB.

Hence chain is lattice.



OBSERVE OPTIMIZE OUTSPREAD