Problems under Ergodic Process:

1. **Let $(t) = A$, where A is a random variable. Prove that $\{(t)\}$ is nota mean ergodic.**

Solution:

Given $(t) = A$, where A is a random variable

To Prove $\{(t)\}$ is a mean ergodic, we have to prove

$$[X(t)] = \lim_{T\to\infty} \overline{X}_T$$

The ensemble mean of $\{(t)\}$ is given by,

$$[X(t)] = E[A] ------(1)$$

The time average is given by,

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt = \frac{1}{2T}\int_{-T}^{T} A\, dt$$

$$= \frac{A}{2T}\int_{-T}^{T} dt = \frac{A}{2T}[t]_{-T}^{T} = \frac{A}{2T}(2T) = A$$

$$\therefore \overline{X}_T = A$$

$$\lim_{T\to\infty} \overline{X}_T = A ----(2)$$

From $(1)nd$ (2)

$$[X(t)] \neq \lim_{T\to\infty} \overline{X}_T$$

$\therefore \{(t)\}$ is not mean Ergodic.

2. A random process has sample functions of the form $X(t) = A\cos(mt + \theta)$, where m is constant and A is a random variable with mean zero and variance one and θ is also a random variable that is uniformly distributed between 0 and 2π. Assume that the random variables A and θ are independent. Prove that X(t) is a mean ergodic process?

Solution:

Given $(t) = A\cos(\omega t + \theta)$, where A is a random variable with mean zero .

$$\therefore (A) = 0, E(A^2) = 1$$

θ is uniformly distributed between 0 and 2π

$$f(\theta) = \frac{1}{2\pi}; 0 < \theta < 2\pi$$

To Prove {X(t)} is Mean Ergodic.

we have to prove

$$[X(t)] = \lim_{T\to\infty} \overline{X}_T$$

The ensemble mean of $\{X(t)\}$ is given by,

$$[X(t)] = E[A\cos(\omega t + \theta)]$$

$$= [A]\cos(\omega t + \theta)\ since\ A\ and\ \theta\ are\ independent\ R.V'S$$

$$= 0 \ldots\ldots\ldots\ldots\ldots..(1)$$

The time average is given by,

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

$$= \frac{1}{2T}\int_{-T}^{T} A\cos(\omega t + \theta)dt$$

$$= \frac{A}{2T}\int_{-T}^{T} \cos(\omega t + \theta)dt$$

$$= \frac{A}{2T}\left[\frac{\sin(\omega t+\theta)}{\omega}\right]_{-T}^{T}$$

$$\overline{X}_T = \frac{}{2T\omega}[\sin(\omega T + \theta) - \sin(-\omega T + \theta)]$$

$$\lim_{T\to\infty} \overline{X}_T = \lim_{T\to\infty} \frac{A}{2T\omega}[\sin(\omega T + \theta) - \sin(-\omega T + \theta)]$$

$$= 0\ldots\ldots\ldots(2)$$

From (1)nd (2) , $E[X(t)] = \lim_{T\to\infty} \overline{X}_T$

$\therefore \{(t)\}$ is a mean Ergodic Process.

Correlation Ergodic Process:

Let {X(t)} be a random process. The ensemble auto correlation function is

$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$

The time auto correlation function is $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} (t)\, X(t + r)\,dt$

A process {X(t)} is said to be correlation ergodic if $R_{XX}(t_1, t_2) = \lim_{T\to\infty} \overline{X}_T$

Problems under Correlation Ergodic Process:

1. **Given a WSS random process $\{(t)\} = 10\cos(100t + \theta)$, where θ is uniformly distributed over $(-\pi, \pi)$. Prove that (t) is a correlation ergodic.**

Solution:

Given $\{(t)\} = 10\cos(100t + \theta)$

$\Rightarrow (\theta) = \frac{1}{2\pi}, -\pi < \theta < \pi$

$R_{XX}(t, t + r) = E[X(t)X(t + r)]$

$= [10\cos(100t+\theta)\ 10cos\ (100(t+r)+\theta)]$

$= \frac{100}{2}[\cos(100t+\theta)\ cos\ (100t+100r+\theta)]$

$= 50\ [\cos(100t+\theta+100t+100r+\theta)+\cos\ (100t+\theta-100t-$

$100r-\theta)]$

$= 50\ [\cos(200t+2\theta+100r)+\cos\ (-100r)]$

$= 50\ [\cos(200t+2\theta+100r)+\cos\ (100r)]$

$= 50\cos\ (100r)+\frac{50}{2\pi}\int_0^{2\pi}\cos\ (200t+100r+2\theta)\theta$

$= 50\cos\ (100r)+\frac{25}{\pi}[\frac{\sin\ (200t+100c+2\theta)}{2}]_0^{2\pi}$

$= 50\cos\ 100r+\frac{25}{2\pi}\ [\sin\ (200t+100r+4\pi)-\sin(200t+100r-0)$

$= 50\cos\ 100r+\frac{25}{2\pi}\ [\sin\ (200t+100r)-\sin(200t+100r)]$

$$R_X(t,t+r\ ) = 50\cos\ 100r$$

Let $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\ X(t+r\ )dt$

$$= \frac{1}{2T}\ [\int_{-T}^{T} 50\ \cos(100r)dt\ +50\ \int_{-T}^{T}\cos(200t+100r+2\theta)dt]$$

$$= \frac{1}{2T}\ \{[50\cos(100r)\ t]_{-T}^{T}+50\,[\frac{\sin\ (200t+100r+2\theta)}{200}]_{T}^{T}\ \}$$

$= \frac{1}{2T}$ $\{[50\cos(100r)\,(T-(-T)] + \frac{50}{200}[\sin(200t+100r+2\theta) -$

$\sin(200(-T)+100r+2\theta)]\}$

$$= \frac{1}{2T}\{[50\cos(100r)\ (2T)]$$

$$+\frac{1}{4}[\sin(200t+100r+2\theta)-\sin(-200T+100r+2\theta)]\}$$

$$\lim_{T\to\infty}\overline{X}_T = \lim_{T\to\infty}\frac{1}{2T}\{[50\cos(100r)\ (2T)]$$

$$+\frac{1}{4}[\sin(200t+100r+2\theta)-\sin(-200T+100r+2\theta)]\}$$

$$= \lim_{T\to\infty}\frac{1}{2T}50\cos(100r)\ (2T) + \lim_{T\to\infty}\frac{1}{8T}[\sin(200t+100r+2\theta)$$

$$-\sin(-200T+100r+2\theta)]$$

$$= \lim_{T\to\infty} 50\cos(100r) + \lim_{T\to\infty}\frac{1}{8T}[\sin(200t+100r+2\theta)$$

$$-\sin(-200T+100r+2\theta)]$$

$$= 50\cos 100r + 0$$

$$= 50\cos 100r$$

$$R_X(t,t+r) = \lim_{T\to\infty}\overline{X}_T$$

Hence (t) is a correlation ergodic.

2. Find the ACF of the periodic time function $(t) = A\ sinmt$.

Solution:

Since periodic time function X(t) is given, we use time auto correlation function.

The ACF of the process is given by $R_X(t_1, t_2) = \lim_{T\to\infty} \overline{X}_T$

To find $\lim_{T\to\infty} \overline{X}_T$

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\, X(t+r)\,dt$$

$$= \frac{1}{2T}\int_{-T}^{T} A\, sin\omega t\; A\sin(\omega t + \omega r)\,dt$$

$$= \frac{A^2}{2T}\int_{-T}^{T} sin\omega t\;\sin(\omega t + \omega r)\,dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T}[\cos(\omega t - \omega t - \omega r) - \cos(\omega t + \omega t + \omega r)]\,dt$$

$$= \frac{A^2}{4T}\int_{-T}^{T}[\cos(-\omega r) - \cos(2\omega t + \omega r)]\,dt$$

$$= \frac{A^2}{4T}\left[cos\omega r(t)_{-T}^{T} - \left(\frac{\sin(2\omega t + \omega r)}{2\omega}\right)_{-T}^{T}\right]$$

$$= \frac{A^2}{4T}[cos\omega(2T) - \frac{\sin(2\omega T + \omega r)}{2\omega} + \frac{\sin(-2\omega T + \omega r)}{2\omega}]$$

$$= \frac{A^2}{4T}\ cos\omega(2T) + \frac{A^2}{4T}[-\frac{\sin(2\omega T + \omega r)}{2\omega} + \frac{\sin(-2\omega T + \omega r)}{2\omega}]$$

The ACF of the process is given by

$$R_X(t_1, t_2) = \lim_{T \to \infty} \overline{X}_T$$

$$= \frac{A^2}{2}\ cos\omega r + \lim_{T \to \infty} \frac{A^2}{4T}[-\frac{\sin(2\omega T + \omega r)}{2\omega} + \frac{\sin(-2\omega T + \omega r)}{2\omega}]$$

$$R_X(t_1, t_2) = \frac{A^2}{2}\ cos\omega r$$