Line, surface and volume integrals

In electromagnetic theory, we come across integrals, which contain vector functions. Some representative integrals are listed below:

$$\int_V \vec{F}\,dv \qquad \int_C \phi\, d\vec{l} \qquad \int_C \vec{F}.d\vec{l} \qquad \int_S \vec{F}.d\vec{s}$$

In the above integrals, $\vec{F}$ and ϕ respectively represent vector and scalar function of space coordinates. *C*,*S* and *V* represent path, surface and volume of integration. All these integrals are evaluated using extension of the usual one-dimensional integral as the limit of a sum, i.e., if a function *f*(*x*) is defined over arrange *a* to *b* of values of *x*, then the integral is given by

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f_i \delta x_i \qquad \text{.................................(1.42)}$$

where the interval (*a,b*) is subdivided into n continuous interval of lengths $\delta x_1, \ldots\ldots\ldots, \delta x_n$.

Line Integral: Line integral $\int_C \vec{E}\cdot\vec{dl}$ is the dot product of a vector with a specified *C*; in other words it is the integral of the tangential component $\vec{E}$ along the curve *C*.

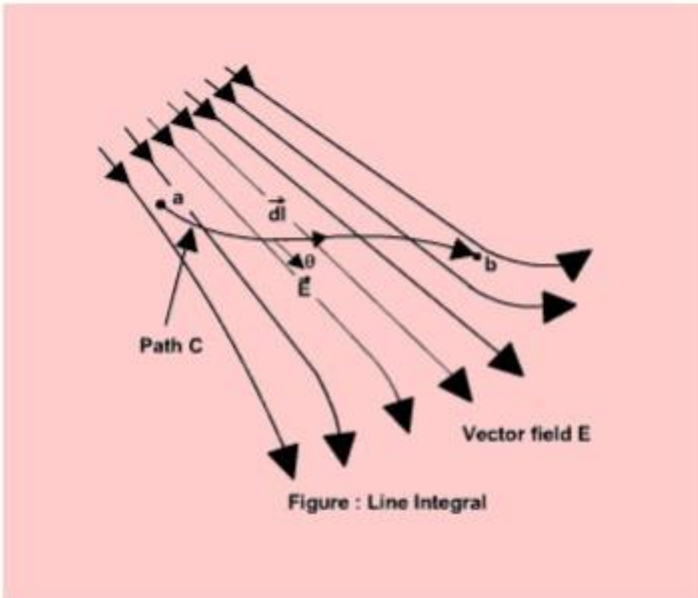

Figure : Line Integral

Fig 1.14: Line Integral

As shown in the figure 1.14, given a vector $\vec{E}$ around *C*, we define the integral $\int_C \vec{E}.\vec{dl} = \int_a^b E\cos\theta dl$ as the line integral of E along the curve C.

If the path of integration is a closed path as shown in the figure the line integral becomes a closed line integral and is called the circulation of $\vec{E}$ around *C* and denoted as $\oint_C \vec{E}.\vec{dl}$ as shown in the figure 1.15.

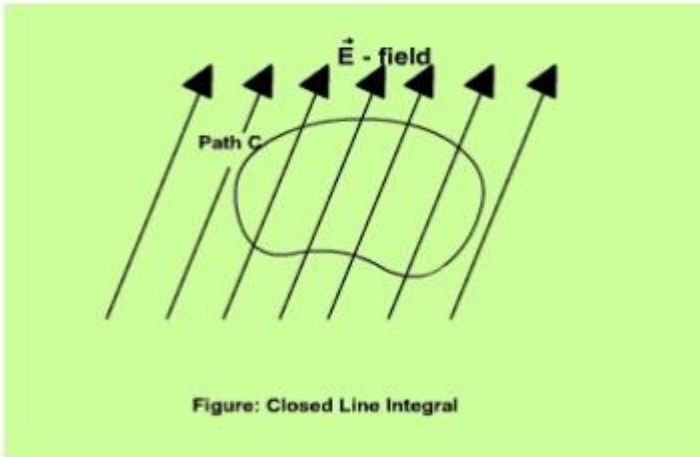

Fig 1.15: Closed Line Integral

Surface Integral :

Given a vector field $\vec{A}$, continuous in a region containing the smooth surface *S*, we define the surface integral or the flux of $\vec{A}$ through *S* as $\psi = \int_S A\cos\theta dS = \int_S \vec{A}.\hat{a}_n dS = \int_S \vec{A} d\vec{S}$ as surface integral over surface S.

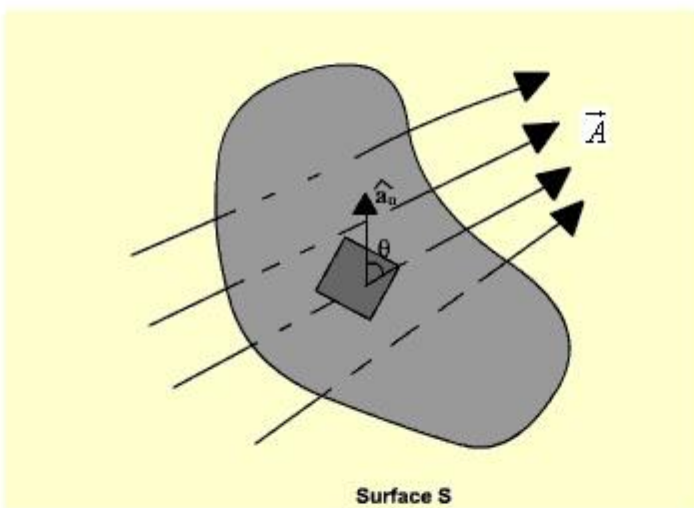

Fig 1.16 : Surface Integral

If the surface integral is carried out over a closed surface, then we write

$$\psi = \oint_S \vec{A} d\vec{S}$$

Volume Integrals:

We define $\int_V f dV$ or $\iiint_V f dV$ as the volume integral of the scalar function *f*(function of spatial coordinates) over the volume *V*. Evaluation of integral of the form $\int_V \vec{F} dV$ can be carried out as a sum of three scalar volume integrals, where each scalar volume integral is a component of the vector $\vec{F}$

The Del Operator :

The vector differential operator ∇ was introduced by Sir W. R. Hamilton and later on developed by P. G. Tait.

Mathematically the vector differential operator can be written in the general form as:

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial}{\partial w}\hat{a}_w \quad \text{.................................(1.43)}$$

In Cartesian coordinates:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \quad \text{..(1.44)}$$

In cylindrical coordinates:

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z \quad \text{..(1.45)}$$

and in spherical polar coordinates:

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi \quad \text{.................................(1.46)}$$

Gradient of a Scalar function:

Let us consider a scalar field V(u,v,w) , a function of space coordinates.

Gradient of the scalar field *V* is a vector that represents both the magnitude and direction of the maximum space rate of increase of this scalar field *V*.

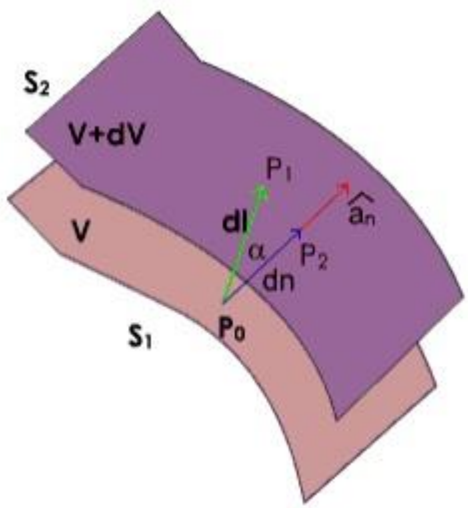

Fig 1.17 : Gradient of a scalar function

As shown in figure 1.17, let us consider two surfaces *S*1and *S*2 where the function *V* has constant magnitude and the magnitude differs by a small amount d*V*. Now as one moves from *S*1 to *S*2, the magnitude of spatial rate of change of *V* i.e. dV/dl depends on the direction of elementary path length dl, the maximum occurs when one traverses from *S*1to *S*2along a path normal to the surfaces as in this case the distance is minimum.

By our definition of gradient we can write:

$$\text{grad}V = \frac{dV}{dn}\hat{a}_n = \nabla V \quad \text{.......(1.47)}$$

since $d\vec{n}$

which represents the distance along the normal is the shortest distance between the two surfaces.

For a general curvilinear coordinate system

$$d\vec{l} = \hat{a}_u\, dl_u + \hat{a}_v\, dl_v + \hat{a}_w\, dl_w = \left(h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w\right) \quad \text{.......(1.48)}$$

Further we can write

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \nabla V \cdot \hat{a}_l \quad \text{.......(1.49)}$$

Hence,

$$dV = \nabla V . dl = \nabla V .(h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w) \quad \text{.......(1.50)}$$

Also we can write,

$$dV = \frac{\partial V}{\partial l_u} dl_u + \frac{\partial V}{\partial l_v} dl_v + \frac{\partial V}{\partial l_w} dl_w$$
$$= \left(\frac{\partial V}{\partial l_u} \hat{a}_u + \frac{\partial V}{\partial l_v} \hat{a}_v + \frac{\partial V}{\partial l_w} \hat{a}_w \right) \cdot \left(dl_u \hat{a}_u + dl_v \hat{a}_v + dl_w \hat{a}_w \right)$$
$$= \left(\frac{\partial V}{h_1 \partial u} \hat{a}_u + \frac{\partial V}{h_2 \partial v} \hat{a}_v + \frac{\partial V}{h_3 \partial w} \hat{a}_w \right) \cdot \left(h_1 du \hat{a}_u + h_2 dv \hat{a}_v + h_3 dw \hat{a}_w \right) \quad \text{.............................(1.51)}$$

By comparison we can write,

$$\nabla V = \frac{1}{h_1} \frac{\partial V}{\partial u} \hat{a}_u + \frac{1}{h_2} \frac{\partial V}{\partial v} \hat{a}_v + \frac{1}{h_3} \frac{\partial V}{\partial w} \hat{a}_w \quad \text{..(1.52)}$$

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for gradient can be written as: In Cartesian coordinates:

$$\nabla V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z \quad \text{..(1.53)}$$

In cylindrical coordinates:

$$\nabla V = \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z \quad \text{...(1.54)}$$

and in spherical polar coordinates:

$$\nabla V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{a}_\phi \quad \text{..(1.55)}$$

The following relationships hold for gradient operator.

$$\nabla(U+V) = \nabla U + \nabla V$$
$$\nabla(UV) = V\nabla U + U\nabla V$$
$$\nabla\left(\frac{U}{V}\right) = \frac{V\nabla U - U\nabla V}{V^2}$$
$$\nabla V^n = nV^{n-1}\nabla V \quad \text{...(1.56)}$$

where *U* and *V* are scalar functions and *n* is an integer.

It may further be noted that since magnitude of $\frac{dV}{dl}(= \Delta V.\hat{a}_1)$ depends on the direction of d*l*, it is called the **directional derivative**. If $A = \Delta V,\ V$ is called the scalar potential function of the vector function $\vec{A}$.