UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.5 NON HOMOGENEOUS LINEAR PDE OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

Non-Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order non-homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2}+a_1\frac{\partial^2 z}{\partial x\partial y}+a_2\frac{\partial^2 z}{\partial y^2}+a_3\frac{\partial z}{\partial x}+a_4\frac{\partial z}{\partial y}=f(x,y) \quad ----- (1)$$

Let the differential operator $D=\frac{\partial}{\partial x}$ & $D'=\frac{\partial}{\partial y}$

$$(1)\Rightarrow \left(D^2+a_1DD'+a_2D'^2+a_3D+a_4D'\right)z=f(x,y) \quad ----- (2)$$

The general solution of equation (2) is

$$\boxed{z=\text{complementary function}+\text{Particular Integral}=\text{C.F+P.I}}$$

To find complementary Function:

Case : I

The given PDE will bring into the form of $(D-m_1D'-C_1)(D-m_2D'-C_2)z=0$

$$\boxed{C.F=e^{c_1x}f_1(y+m_1x)+e^{c_2x}f_2(y+m_2x)}$$

Case : II

The given PDE will bring into the form of $(D-mD'-C)^2z=0$

$$\boxed{C.F=e^{cx}f_1(y+mx)+xe^{cx}f_2(y+mx)}$$

Note: Particular Integral can be obtained, similar like in Homogeneous types.

1. **Solve** $\left(D^2+2DD'+D'^2-2D-2D'\right)z=e^{3x+y}+\sin(x+2y)$

Solution:

Given $\left(D^2+2DD'+D'^2-2D-2D'\right)z=e^{3x+y}+\sin(x+2y)$

To find C.F

$$\left((D+D')^2-2(D+D')\right)z=0$$

$$(D+D')(D+D'-2)z=0 \quad ------(1)$$

This is of the form

$$(D-m_1D'-C_1)(D-m_2D'-C_2)z=0------(2)$$

Comparing (1) & (2)

$$\boxed{m_1=-1,C_1=0,m_2=-1,C_2=2}.$$

$$C.F=e^{c_1x}f_1(y+m_1x)+e^{c_2x}f_2(y+m_2x)$$

$$\boxed{C.F=f_1(y-x)+e^{2x}f_2(y-x)} \qquad \because e^0=1$$

To find P.I

$$P.I=\frac{1}{D^2+2DD'+D'^2-2D-2D'}\left(e^{3x+y}+\sin(x+2y)\right)$$

	$P.I = P.I_1 + P.I_2$ **To find P.I₁** $P.I_1 = \frac{1}{D^2+2DD'+D'^2-2D-2D'} e^{3x+y}$ Rule: replace $D=3$ & $D'=1$ Type:1 $= \frac{1}{9+6+1-6-2} e^{3x+y}$ $\boxed{P.I_1 = \frac{1}{8} e^{3x+y}}$ **To find P.I₂** $P.I_2 = \frac{1}{D^2+2DD'+D'^2-2D-2D'} \sin(x+2y)$ here $a=1, b=2$ type:2 Rule: replace $D^2 = -(a^2) = -1; D'^2 = -(b^2) = -4$ & $DD' = -(ab) = -2$ $P.I_2 = \frac{1}{-1+2(-2)-4-2D-2D'} \sin(x+2y) = \frac{1}{-1-4-4-2D-2D'} \sin(x+2y)$ $= \frac{-1}{2D+2D'+9} \times \frac{D}{D} \sin(x+2y)$ $= \frac{-D}{2D^2+2D'^2+9D} \sin(x+2y)$ $= \frac{-D}{-2-8+9D} \sin(x+2y) = \frac{-D}{9D-10} \sin(x+2y)$ If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constent term and multiplied with both Nr. & Dr. $= \frac{-D}{9D-10} \times \frac{9D+10}{9D+10} \sin(x+2y)$ $= \frac{-9D^2-10}{81D^2-100} \sin(x+2y)$ $= \frac{-9D^2 \sin(x+2y) - 10D\sin(x+2y)}{-81-100}$ $= \frac{1}{-181}\left[-9D\cos(x+2y) - 10\cos(x+2y)\right]$ $\boxed{P.I_2 = \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]}$ $\boxed{P.I = \frac{1}{8} e^{3x+y} + \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]}$ **The general solution is** $z = C.F + P.I$ $\boxed{z = f_1(y-x) + e^{2x} f_2(y-x) + \frac{1}{8} e^{3x+y} + \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]}$
2.	Solve $\left(D^2 - D'^2 - 3D + 3D'\right) z = e^{3x+y} + 4$ **Solution:** Given $\left(D^2 - D'^2 - 3D + 3D'\right) z = e^{3x+y} + 4$ **To find C.F** $\left((D+D')(D-D') - 3(D-D')\right) z = 0$ $(D-D')(D+D'-3) z = 0 \quad -----(1)$ This is of the form $(D - m_1 D' - C_1)(D - m_2 D' - C_2) z = 0 -----(2)$ Comparing (1) & (2)

	$m_1 = 1, C_1 = 0, m_2 = -1, C_2 = 3.$ $C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$ $C.F = f_1(y+x) + e^{3x} f_2(y-x)$ $\quad \because e^0 = 1$ **To find P.I** $P.I = \frac{1}{D^2 - D'^2 - 3D + 3D'}\left(e^{3x+y} + 4\right)$ $P.I = P.I_1 + P.I_2$ $P.I_1 = \frac{1}{D^2 - D'^2 - 3D + 3D'} e^{3x+y}$ $\quad$ here $a = 3, b = 1$ $\quad$ type : 1 $= \frac{1}{9 - 1 - 9 + 3} e^{3x+y}$ $\quad$ Rule: Replace $D = 3, D' = 1$ $P.I_1 = \frac{1}{2} e^{3x+y}$ $P.I_2 = \frac{1}{D^2 - D'^2 - 3D + 3D'} 4e^{0x+0y}$ $\quad$ here $a = 0, b = 0$ $\quad$ type : 1 $= \frac{1}{0} 4e^{0x+0y}$ $\quad$ Rule: Replace $D = 0, D' = 0$ Introduce x in Nr. and Diff. Dr. Partially w.r.to.D in the previous step $= \frac{x}{2D - 0 - 3 + 0} 4e^{0x+0y}$ $= \frac{4x}{-3}$ $P.I_2 = \frac{-4x}{3}$ $P.I = \frac{1}{2} e^{3x+y} - \frac{4x}{3}$ The general solution is $z = C.F + P.I$ $z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{1}{2} e^{3x+y} - \frac{4x}{3}$
3.	**Solve** $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right) z = xe^y$ **Solution:** Given $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right) z = xe^y$ **To find C.F** $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right) z = 0$ $(2D + D')(D - D') + 3(2D + D') z = 0$ $(2D + D')(D - D' + 3) z = 0$ $\left(D + \frac{D'}{2}\right)(D - D' + 3) z = 0$ This is of the form $(D - m_1 D' - C_1)(D - m_2 D' - C_2) z = 0 - - - - - (2)$ Comparing (1) & (2) $m_1 = \frac{-1}{2}, C_1 = 0, m_2 = 1, C_2 = -3.$

$C.F = e^{c_1 x} f_1(y+m_1 x) + e^{c_2 x} f_2(y+m_2 x)$

$$\boxed{C.F = f_1\left(y-\frac{x}{2}\right) + e^{-3x} f_2(y+x) \qquad \because e^0 = 1}$$

To find P.I

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^y \qquad \text{Type}:4$$

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^{0x+y} \qquad \text{here } a = 0, b = 1$$

Rule: replace $D = D + a = D + 0 = D$; $D' = D' + b = D' + 1$

$$P.I = e^{0x+y} \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)} x$$

$$= e^y \frac{1}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} x \qquad \text{Type}:3$$

$$= e^y \frac{1}{2D^2 - DD' + 5D - D'^2 + D' + 2} x$$

$$= e^y \frac{1}{2\left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right)\right]} x$$

[normally we take out highest power term of D in the homogeneous type, but it is not necessary in the non-homogeneous type]

$$= \frac{e^y}{2}\left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right)\right]^{-1} x \qquad \because D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

$$= \frac{e^y}{2}\left[1 - \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right) + \ldots\right] x \quad \text{[neglect the terms D}', \text{ since } D'(x) = 0]$$

$$= \frac{e^y}{2}\left[1 - \left(\frac{5D}{2}\right) + \ldots\right] x \qquad \because D^2(x) = 0$$

$$= \frac{e^y}{2}\left[x - \frac{5(1)}{2}\right] \qquad \because D^2(x) = 0$$

$$= \frac{e^y}{2}\left[\frac{2x-5}{2}\right]$$

$$\boxed{P.I = \frac{e^y}{4}[2x-5]}$$

The general solution is

$z = C.F + P.I$

$$\boxed{z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{e^y}{4}[2x-5]}$$