DESIGN OF IIR FILTER FROM ANALOG FILTER

ANALOG FILTER DESIGN – BUTTERWORTH FILTER

The popular methods of designing IIR digital filter involve the design of equivalent analog filter and then converting the analog filter to digital filter. Hence to design a Butterworth IIR digital filter, first an analog Butterworth filter transfer function is determined using the given specifications. Then the analog filter transfer function is converted to a digital filter transfer function by using either impulse invariant transformation or bilinear transformation.

The filter passes all frequencies below Ωc. This is called pass band of the filter. Also the filter blocks all the frequencies above Ωc. This is called stop band of the filter. Ωc is called cutoff frequency or critical frequency.

No Practical filters can provide the ideal characteristic. Hence approximations of the ideal characteristic are used. Such approximations are standard and used for filter design. Butterworth filters are defined by the property that the magnitude response is maximally flat in the pass band.

Let ω_p = pass band edge digital frequency in rad/sample.

ω_s = Stop band edge digital frequency in rad/sample

$T = \frac{1}{F_s}$ = Sampling time in sec

Where F_s = Sampling frequency in Hz.

A_p = Gain at a pass band frequency ω_p

A_s = Gain at a pass band frequency ω_s

1. Choose either bilinear or impulse invariant transformation, and determine the specifications of analog filter. The gain or attenuation of analog filter is same as digital filter. The band edge frequencies are calculated using the following equations.

Let Ω_p = Pass band edge analog frequency corresponding to ω_p

Ω_s = Stop band edge analog frequency corresponding to ω_s

For bilinear transformation

$$\Omega p = \frac{2}{T} \tan \frac{\omega_p}{2}$$

$$\Omega s = \frac{2}{T} \tan \frac{\omega_s}{2}$$

For impulse invariant transformation

$$\Omega p = \frac{\omega_p}{T}$$

$$\Omega s = \frac{\omega_s}{T}$$

2. Decide the order N of the filter. In order to estimate the order N, calculate the parameter N_1 using the following equation

$$N_1 = \frac{1}{2} \frac{\log\left[\frac{(1/A_s^2)-1}{(1/A_p^2)-1}\right]}{\log\frac{\Omega s}{\Omega p}}$$

Choose N such that≥N_1.Usually N is chosen as nearest integer just greater than N_1.

3. Determine the normalized transfer function, $H(s_n)$ of the analog low pass filter.

When N is even,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

When N is odd

$$H(s_n) = \frac{1}{s_n+1} \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Where $b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$

4. Calculate the analog cutoff frequency, Ω c.

Cutoff frequency, $\Omega c. = \frac{\Omega s}{\left[\left(\frac{1}{A_s{}^2}\right)-1\right]^{\frac{1}{2N}}}$

5. Determine the un normalized analog transfer function H(s) of the low pass filter.

$$H(s) = H(s_n)|_{sn = s/\Omega c}$$

When the order N is even, H(s) is obtained by letting $s_n \rightarrow s/\Omega c$

$$H(s) = \prod_{k=1}^{N/2} \frac{1}{s_n^2 + b_k s_n + 1} \Big|_{sn = s/\Omega c}$$

$$= \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

When the order N is odd, H(s) is obtained by letting $sn = s/\Omega c$

In above equation

$$H(s) = \frac{1}{s_n+1} \prod_{k=1}^{N-1/2} \frac{1}{s_n^2 + b_k s_n + 1} \Big|_{sn = s/\Omega c}$$

$$= \frac{\Omega c}{s+\Omega c} \prod_{k=1}^{N-1/2} \frac{1}{s^2 + s\, b_k \Omega c + \Omega c^2}$$

6.Determine the transfer function of digital filter, H(z).Using the chosen transformation in step-1 transform H(s) into H(Z).When impulse invariant transformation is employed ,if T<1,then multiply H(Z) by T to normalize the magnitude.

7. Realize the digital filter transfer function H(z) by a suitable structure.

8. Verify the design by sketching the frequency response $H(e^{j\omega})$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

The basic filter design is low pass filter design. The high pass, band pass or band stop filters are obtained from low pass filter design by frequency transformation.

ANALOG FILTER DESIGN –CHEBYSHEV FILTER

Chebyshev filters are analog or digital filters having a steeper roll-off than Butterworth filters, and have pass band ripple (type I) or stop band ripple (type II). Chebyshev filters have the property that they minimize the error between the idealized and the actual filter characteristic over the range of the filter, but with ripples in the pass band. This type of filter is named after Pafnuty Chebyshev because its mathematical characteristics are derived from Chebyshev polynomials. Type I Chebyshev filters are usually referred to as "Chebyshev filters", while type II filters are usually called "inverse Chebyshev filters".

DESIGN PROCEDURE FOR LOWPASS DIGITAL CHEBYSHEV IIR FILTER

Let ω_p = Pass band edge digital frequency in rad/sample

ω_s = Stop band edge digital frequency in rad/sample

$T=1/F_s$ =Sampling time in seconds

Where F_s = Sampling frequency in Hz

A_P =Gain at a pass band frequency ωp

A_s =Gain at a Stop band frequency ω_s

Step-1:

Choose either bilinear or impulse invariant transformation, and determine the specifications of equivalent analog filter. The gain and attenuation of analog filter is same as digital filter. the band edge frequencies are calculated using the following equations.

Let Ωp= Pass band edge analog frequency corresponding to ω_p

Ω_s= Stop band edge analog frequency corresponding to ω_p

For bilinear transformation

$$\Omega_p = \frac{2}{T}\tan\frac{\omega p}{2}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega s}{2}$$

For impulse invariant transformation

$$\Omega_p = \frac{\omega_p}{T}$$

$$\Omega_s = \frac{\omega_s}{T}$$

Step-2

Decide the order N of the filter .In order to estimate the order N calculate the parameter N_1 using the following equation.Choose N such that N>= N_1.Usually N is chosen as nearest integer just greater than N_1.

$$N_1 = \frac{cosh^{-1}\left[\left(\frac{(1/A_S^2)-1}{1/A_p^2)-1}\right)^{\frac{1}{2}}\right]}{cosh^{-1}\frac{\Omega s}{\Omega p}}$$

Step-3:

Determine the normalized transfer function $H(s_n)$ of the filter

$$H(s_n) = \prod_{k=`1}^{\frac{N}{2}} \frac{B_k}{s_n^2 + b_k s_n + c_k}$$

When the order N is odd

$$H(s) = \frac{B_0}{s+C_0} \prod_{k=`1}^{\frac{N-1}{2}} \frac{B_k}{s^2 + b_k s + c_k}$$

Where

$$b_k = 2y_N \sin\left(\frac{(2k-1)\pi}{2N}\right)$$

$$c_k = y_N^2 + cos^2\left(\frac{(2k-1)\pi}{2N}\right)$$

$$c_0 = y_N$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{e^2}+1\right)^{\frac{1}{2}} + \frac{1}{e}\right]^{\frac{1}{N}}\right\} - \frac{1}{2}\left\{\left[\left(\frac{1}{e^2}+1\right)^{\frac{1}{2}} + \frac{1}{e}\right]^{\frac{1}{N}}\right\}$$

$$\epsilon = \left[\left(\frac{1}{A_P^2}\right) - 1\right]^{\frac{1}{2}}$$

For even values of N,find B_k such that

H (0)=1

(It is normal practice to take $B_0 = B_1 = B_2 ..... = B_k$)

4. Determine the un normalized analog transfer function H(s) of the low pass filter

$H(s)=H(s_n) \mid_{S_n} = \frac{s}{\Omega c}$

Here $\Omega_{c=}\ \Omega_p$ =Pass band edge frequency

When the order N is even, H (s) is obtained by letting $S_n = \frac{s}{\Omega c}$

$$H(s)= \prod_{k=1}^{\frac{N}{2}} \frac{B_k}{s_n^2 + b_k s_n + c_k} \Big| \; S_n = \frac{s}{\Omega c}$$

$$= \prod_{k=1}^{\frac{N}{2}} \frac{B_k \, \Omega c^2}{s^2 + b_k \Omega c s + c_k \, \Omega c^2}$$

When the order N is odd H(s) is obtained by letting $S_n \rightarrow \frac{s}{\Omega c}$

$$H(s) = \frac{B_0}{s+C_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s^2 + b_k s + c_k} \Big| \; S_n \rightarrow \frac{s}{\Omega c}$$

5.Determine the transfer function of digital filter, H(z).Using the chosen transformation in step-1 transform H(s) into H(z).When impulse invariant transformation is employed, if T<1 then multiply H(z) by T to normalize the magnitude.

6. Realize the digital filter transfer function H(z) by a suitable structure.

7. Verify the design by sketching the frequency response. $H(e^{j\omega})$

$$H(e^{j\omega}) = H(z) \,|_{z= e^{j\omega}}$$

The high pass, band pass and band stop filters are obtained from low pass filter design by frequency transformation.

1. Design an analog butter worth filter that has a -2dB pass band attenuation at a frequency of 20 rad/sec and at least -10dB stop band attenuation at 30 rad/sec.

Given: $A_p = 2$ dB

$A_s = 10$ dB

$\Omega_P = 20$ rad/sec

$\Omega_s = 30$ rad / sec

$$N \geq \frac{\log\sqrt{\left[\frac{10^{0.1\,As}-1}{10^{0.1\,Ap}-1}\right]}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\sqrt{\left[\frac{10-1}{10^{0.2}-1}\right]}}{\log\left(\frac{30}{20}\right)}$$

$$\geq \frac{o.5935}{0.1760}$$

$$N \geq 3.372$$

$$N = 4$$

$$N = 4, \text{ so } H(s) = \frac{1}{(s^2 + o.76537s+1)(s^2 + 1.8477s+1)}$$

Now find cut off frequency Ω_c

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\,Ap} - 1\right)^{1/2N}}$$

$$= \frac{20}{(10^{0.1*2} - 1)^{1/8}}$$

$$\Omega_c = \frac{20}{(0.5848)^{1/8}} = \frac{20}{0.9351}$$

$$\Omega_c = 21.3872$$

$$s \rightarrow s / \Omega_c \text{ in } H(s)$$

$$H(s) = \frac{1}{\left[\left(\frac{s}{21.387}\right)^2 + 0.76537\left(\frac{s}{21.387}\right)+1\right]\left[\left(\frac{s}{21.387}\right)^2 + 0.76537\left(\frac{s}{21.387}\right)+1\right]}$$

$$H(s) = \frac{1}{\left[\frac{(s^2 + 16.3689s+457.40)}{457.40}\right]\left[\frac{(s^2 + 39.5167s+457.4)}{457.40}\right]}$$

$$H(s) = \frac{0.20921 * 10^6}{(s^2 + 16.3689s+457.4)(s^2 + 16.3689s+457.4)}$$

2. **Obtain an analog chebyshev filter transfer function that satisfies the constrains.**

$0.707 \leq |H(j\Omega)| \leq 1 \quad 0 \leq \Omega \leq 2$

$|H(j\Omega)| \leq 0.1 \quad \Omega \geq 4$

Step1: Find the order of filter N

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707,$$

$$\frac{1}{0.707} = \sqrt{1+\varepsilon^2}$$

$1+\varepsilon^2 = 2 \qquad \varepsilon^2 = 2-1 = 1 \qquad \varepsilon = 1$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.1$$

$$\frac{1}{0.1} = \sqrt{1+\lambda^2}$$

$1+\lambda^2 = 10^2$

$\lambda^2 = 100-1 = 99 \qquad \lambda = 9.949$

To Find N:

$$N \geq \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1}\left(\Omega_s \middle| \Omega_p\right)}$$

$$N \geq \frac{\cosh^{-1}(\frac{9.949}{1})}{\cosh^{-1}(\frac{4}{2})}$$

$$\geq \frac{2.988}{1.316} \geq 2.2688$$

Step 2: By rounding N = 3

Step 3: Find the value of a and b:

$$a = \Omega_p \left(\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right)$$

$$b = \Omega_p \left(\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right)$$

$$\varepsilon = \sqrt{10^{0.1\,Ap}-1}$$

$$\varepsilon = \sqrt{10^{0.1\,X3}-1} = 0.9976$$

$$\varepsilon = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}}$$

$$= 1 + \sqrt{1 + 1} = 1 + 1.414 = 2.414$$

$$\mu = 2.414$$

$$a = \Omega_p \left(\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right)$$

$$\mu = 1 + \sqrt{1 + 1} = 2.414$$

$$a = \Omega_p \left(\frac{\mu^{1} - \mu^{-1}}{2} \right)$$

$$= 2\left(\frac{2.414^{\frac{1}{3}} - 2.414^{\frac{-1}{3}}}{2} \right)$$

$$a = 0.596$$

$$b = \Omega_p \left(\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right)$$

$$= 2\left(\frac{2.414^{\frac{1}{3}} + 2.414^{\frac{-1}{3}}}{2}\right)$$

b = 2.0869

Step 4: Calculating poles of chebyshev filter.

$$\Phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2,3$$

$$\Phi_1 = \frac{\pi}{2} + \frac{\pi}{6} \quad = 120°$$

$$\Phi_2 = \frac{\pi}{2} + \frac{3\pi}{6} \quad = 180°$$

$$\Phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} \quad = 240°$$

We know that

$$S_K = a \cos \Phi_k + j\, b \sin \Phi_k \qquad k = 1, 2, 3$$

$$s_1 = a \cos\Phi_1 + j\, b \sin \Phi_1$$

$$= 0.596 \cos 120 + j\, 2.0869 \sin 120$$

$$s_1 = -0.298 + j\, 1.8073$$

$$s_2 = 0.596 \cos 180 + j\, 2.0869 \sin 180$$

$$s_2 = -0.596$$

$$s_3 = 0.596 \cos 240 + j\, 2.0869 \sin 240$$

$$s_3 = -0.298 - j\, 1.8073$$

Step 5: Find denominator polynomial of the transfer function using poles.

$$H(s) = (S-S_1)\,(S-S_2)\,(S-S_3)$$

$$= (S + 0.298 - j\, 1.8073)\,(S + 0.596)\,(S + 0.298 + j\, 1.8073)$$

$$= (S + 0.596)\,((S + 0.298)^2 + (1.8073)^2)$$

$$H(s) = (S + 0.596)\,(S^2 + 0.088 + 0.596\, S + 3.2663)$$

$$H(s) = (S + 0.596)\,(S^2 + 0.596\, S + 3.355)$$

Step 6: Find the numerator S=0 in the denominator.

Numerator = S → 0 in the denominator

= 0.596 (3.355)

= 1.999

Then,

$$H(s) = \frac{1.999}{(S + 0.596)\ (S2 + 0.596\ S + 3.355)}$$