5.3 IMPULSE RESPONSE

When the input to a discrete time system is a unit impulse $\delta(n)$ then the output is called an impulse response of the system and is denoted by $h(n)$

∴Impulse response $h(n)$ = H{$\delta(n)$}

$$\delta(n) \rightarrow \boxed{H} \rightarrow h(n)$$

Impulse response of interconnected systems

Parallel connections of discrete time systems (Distributive property)

Consider two LTI systems with impulse response h1(n) and h2(n) connected in parallel as shown in Fig

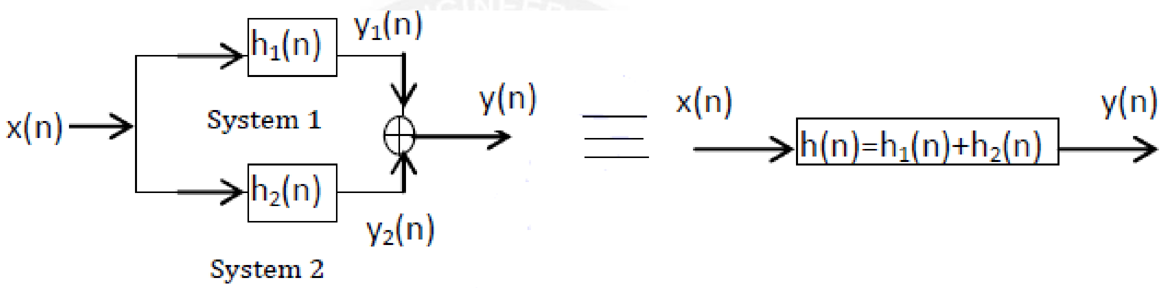
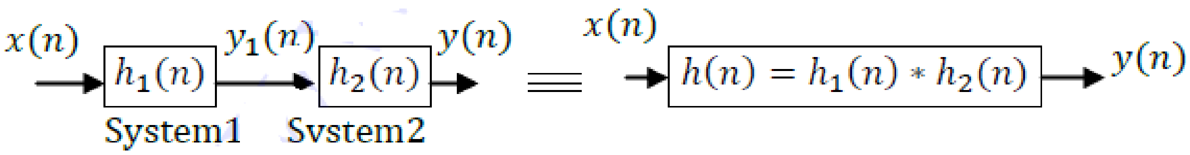


Fig Parallel connections of discrete time systems

Cascade connection of discrete time systems (Associative property)



Let us consider two systems with impulse $h_1(n)$ and $h_2(n)$ connected in cascade as shown in Fig

Example 1 : Determine the frequency response and impulse response

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

Solution:

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

Applying DTFT

$$Y(e^{j\omega}) - \frac{1}{6}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

Frequency Response

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-2j\omega}} = \frac{e^{2j\omega}}{e^{2j\omega} - \frac{1}{6}e^{j\omega} - \frac{1}{6}}$$

$$\frac{H(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{e^{2j\omega} - \frac{1}{6}e^{j\omega} - \frac{1}{6}} = \frac{A}{e^{j\omega} - \frac{1}{2}} + \frac{B}{e^{j\omega} + \frac{1}{3}}$$

$$e^{j\omega} = A\left(e^{j\omega} + \frac{1}{3}\right) + B\left(e^{j\omega} - \frac{1}{2}\right)$$

At $e^{j\omega} = -\frac{1}{3}$

$-\frac{1}{3} = B\left(-\frac{1}{3} - \frac{1}{2}\right)$, $\therefore B = \frac{2}{5}$

At $e^{j\omega} = \frac{1}{2}$

$\frac{1}{2} = A\left(\frac{1}{2} + \frac{1}{3}\right)$, $\therefore A = \frac{3}{5}$

$$H(e^{j\omega}) = \frac{\frac{3}{5}e^{j\omega}}{e^{j\omega} - \frac{1}{2}} + \frac{\frac{2}{5}e^{j\omega}}{e^{j\omega} + \frac{1}{3}}$$

Applying Inverse DTFT,

$$h(n) = \frac{3}{5}\left(\frac{1}{2}\right)^n u(n) + \frac{2}{5}\left(-\frac{1}{3}\right)^n u(n)$$

Example 2:Find response of system using DTFT.

$$h(n) = \left(\frac{1}{2}\right)^n u(n), x(n) = \left(\frac{3}{4}\right)^n u(n).$$

Solution:

$$h(n) = \left(\frac{1}{2}\right)^n u(n), x(n) = \left(\frac{3}{4}\right)^n u(n).$$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \quad ; \quad X(e^{j\omega}) = \frac{1}{1 - \frac{3}{4}e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{3}{4}e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-\frac{1}{2}} \cdot \frac{e^{j\omega}}{e^{j\omega}-\frac{3}{4}}$$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{\left(e^{j\omega}-\frac{1}{2}\right)\left(e^{j\omega}-\frac{3}{4}\right)} = \frac{A}{e^{j\omega}-\frac{1}{2}} + \frac{B}{e^{j\omega}-\frac{3}{4}}$$

$$e^{j\omega} = A\left(e^{j\omega}-\frac{3}{4}\right) + B\left(e^{j\omega}-\frac{1}{2}\right)$$

At $e^{j\omega} = \frac{1}{2}$

$\frac{1}{2} = A\left(\frac{1}{2}-\frac{3}{4}\right)$, $\therefore A = -2$

At $e^{j\omega} = \frac{3}{4}$

$\frac{3}{4} = B\left(\frac{3}{4}-\frac{1}{2}\right)$, $\therefore B = 3$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{-2}{e^{j\omega}-\frac{1}{2}} + \frac{3}{e^{j\omega}-\frac{3}{4}} \quad => \quad Y(e^{j\omega}) = \frac{-2e^{j\omega}}{e^{j\omega}-\frac{1}{2}} + \frac{3e^{j\omega}}{e^{j\omega}-\frac{3}{4}}$$

Applying DTFT,

$$y(n) = -2\left(\frac{1}{2}\right)^n u(n) + 3\left(\frac{3}{4}\right)^n u(n)$$

5.5 LTI SYSTEM ANALYSIS USING DTFT

Output of LTI system is given by linear convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Let the system be excited by the sinusoidal or phaser $e^{j\omega n}$.

$$\therefore x(n) = e^{j\omega n} \, for -\infty < n < \infty$$

Hence the signal is complex in nature .It has unit amplitude and frequency is $'\omega'$.The output is given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)}$$

$$== \sum_{k=-\infty}^{\infty} h(k)e^{j\omega n}.e^{j\omega k}$$

$$= \left[\sum_{k=-\infty}^{\infty} h(k)e^{j\omega k}\right] e^{j\omega n}$$

$$= H(\omega)e^{j\omega n}$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega k}$$

$H(\omega)$ is the Fourier transform of $h(k)$ and $h(k)$ is the unit sample response. $H(\omega)$ is called the transfer function of the system. $H(\omega)$ is complex valued function of ω in the range $-\pi \leq \omega \leq \pi$.The transfer function of $H(\omega)$ can be expressed in polar form as

$$H(\omega) = |H(\omega)|e^{j \llcorner H(\omega)}$$

$$|H(\omega)| is\ the\ magnitude\ of\ H(\omega)$$

$$\llcorner H(\omega) is\ the\ angle\ of\ H(\omega)$$

LTI SYSTEM ANALYSIS USING Z-TRANSFORM

The Z-Transform of impulse response is called transfer or system function H(Z).

$$Y(Z) = X(Z)H(Z)$$

General form of LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Computing the Z-Transform

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Example 1: Consider the system described by the difference equation.

$$y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{5}{4}y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{16}y[n-3]$$

Solution:

$$y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{5}{4}y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{16}y[n-3]$$

Here $N=3,\ M=1$. Order 3 homogeneous equation:

$$y[n]-\frac{5}{4}y[n-1]+\frac{1}{2}y[n-2]-\frac{1}{16}y[n-3]=0 \qquad n\geq 2$$

The characteristic equation:

$$1-\frac{5}{4}a^{-1}+\frac{1}{2}a^{-2}-\frac{1}{16}a^{-3}=0$$

The roots of this third order polynomial is: $a_1=a_2=1/2 \quad a_3=1/4$ and

$$y_h[n]=h[n]=A_1(\frac{1}{2})^n+A_2n(\frac{1}{2})^n+A_3(\frac{1}{4})^n, \quad n\geq 2$$

Let us assume $y[-1]=0$ then (3.52) for this case becomes:

$$\begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \end{bmatrix}\cdot\begin{bmatrix} y[0] \\ y[1] \end{bmatrix}=\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 \\ -5/4 & 1 \end{bmatrix}\cdot\begin{bmatrix} y[0] \\ y[1] \end{bmatrix}=\begin{bmatrix} 1 \\ 1/3 \end{bmatrix} \quad \Rightarrow y[0]=1;\ y[1]=19/12$$

with these we have the impulse response of this system:

$$h[n]=-\frac{4}{3}(\frac{1}{2})^n+\frac{10}{3}n(\frac{1}{2})^n+\frac{7}{3}(\frac{1}{4})^n, \quad n\geq 0$$