UNIT –III

SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

3.1 FIXED POINT ITERATION x=g(x) method

The method is fixed –point iteration (we also call it the $x = g(x)$ method) is a very useful method to get a root of $f(x) = 0$.

Suppose, we want the approximate roots of the equation $f(x) = 0$, we rearrange $f(x)$ into an equivalent form $x = g(x) \ldots.(1)$.

Assume, x_0 to be the starting approximate value to the actual root $r.(ie).,f(r) = 0$.

Setting $x = x_0$ in the right hand side of (1), we get first approximation $x_1 = g(x_0)$

Again setting $x = x_1$ on the R.H.S of (1), we get successive approximations

$$x_2 = g(x_1)$$

$$x_3 = g(x_2)$$

$$x_4 = g(x_3)$$

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$$x_n = g(x_{n-1})$$

The sequence of approximate roots $x_1, x_2, x_3, \ldots. x_n$, if it converges to r is taken as the root of the equation $f(x) = 0$. Whenever, we have $x_r = g(x_r), r$ is said to be a fixed point for the function g.

The condition for the convergence of the method

Theorem

Let $f(x) = 0$ be the given equation whose actual root is r. The equation $f(x) = 0$ be written as $x = g(x)$. Let I be the interval containing the root $x = r$. If $|g'(x)| < 1$ for all x in I, then the sequence of approximations $x_0, x_1, x_2, x_3, \ldots. x_n$ will converge to r, if the initial starting value x_0 is chosen in I.

Example:

1. Find the real root of the equation $x^3 + x^2 - 100 = 0$.

Solution:

Given, $x^3 + x^2 - 100 = 0$

Let $f(x) = x^3 + x^2 - 100$

$f(0) = -100.$

$f(1) = 1 + 1 - 100 = -98 = -ive.$

$f(2) = 8 + 4 - 100 = -98 = -ive.$

$f(3) = 27 + 9 - 100 = -64 = -ive.$

$f(4) = 64 + 16 - 100 = -20 = -ive.$

$f(5) = 125 + 25 - 100 = +50 = +ive.$

So, a root lies between 4 and 5.

The given equation can be written as

$x^2(x + 1) = 100$

$$x^2 = \frac{100}{x+1}$$

$$x = \frac{10}{\sqrt{x+1}} = g(x)$$

$$g(x) = \frac{10}{\sqrt{x+1}}$$

$$g'(x) = 10\,\frac{\left(\frac{-1}{2}\right)}{(x+1)^{3/2}} = \frac{-5}{(x+1)^{3/2}}$$

$$|g'(x)| = \frac{5}{(x+1)^{3/2}}$$

$$|g'(4)| = \frac{5}{5^{3/2}} < 1$$

$$|g'(5)| = \frac{5}{6^{3/2}} < 1$$

$|g'(x)|$ is less than 1 in the interval $(4,5)$

So, this method can be applied.

Let, $x_0 = 4.2$

$$x_1 = g(x_0) = \frac{10}{\sqrt{x_0 + 1}} = \frac{10}{\sqrt{4.2 + 1}} = 4.38529$$

$$x_2 = g(x_1) = \frac{10}{\sqrt{x_1 + 1}} = \frac{10}{\sqrt{4.38529 + 1}} = 4.30919$$

$$x_3 = g(x_2) = \frac{10}{\sqrt{x_2 + 1}} = \frac{10}{\sqrt{4.30919 + 1}} = 4.33996$$

$$x_4 = g(x_3) = \frac{10}{\sqrt{x_3 + 1}} = \frac{10}{\sqrt{4.33996 + 1}} = 4.32744$$

$$x_5 = g(x_4) = \frac{10}{\sqrt{x_4 + 1}} = \frac{10}{\sqrt{4.32744 + 1}} = 4.33252$$

$$x_6 = g(x_5) = \frac{10}{\sqrt{x_5 + 1}} = \frac{10}{\sqrt{4.33252 + 1}} = 4.33046$$

$$x_7 = g(x_6) = \frac{10}{\sqrt{x_6 + 1}} = \frac{10}{\sqrt{4.33046 + 1}} = 4.33129$$

$$x_8 = g(x_7) = \frac{10}{\sqrt{x_7 + 1}} = \frac{10}{\sqrt{4.33129 + 1}} = 4.33096$$

$$x_9 = g(x_8) = \frac{10}{\sqrt{x_8 + 1}} = \frac{10}{\sqrt{4.33096 + 1}} = 4.33109$$

$$x_{10} = g(x_9) = \frac{10}{\sqrt{x_9 + 1}} = \frac{10}{\sqrt{4.33109 + 1}} = 4.33104$$

$$x_{11} = g(x_{10}) = \frac{10}{\sqrt{x_{10} + 1}} = \frac{10}{\sqrt{4.33104 + 1}} = 4.33106$$

$$x_{12} = g(x_{11}) = \frac{10}{\sqrt{x_{11} + 1}} = \frac{10}{\sqrt{4.33106 + 1}} = 4.33105$$

$$x_{13} = g(x_{12}) = \frac{10}{\sqrt{x_{12} + 1}} = \frac{10}{\sqrt{4.33105 + 1}} = 4.33105$$

Here $x_{12} = x_{13} = 4.33105$ correct to 5 decimal places.

Hence, the better approximate root is 4.33105.

Example :

Find the real root of the equation $\cos x = 3x - 1$, correct to 5 decimal places by fixed point iteration method.

Solution:

Given, $\cos x = 3x - 1$

Let $f(x) = \cos x = 3x - 1$

$f(0) = 1 - 0 + 1 = 2 = +ive$.

$f(1) = \cos 1 - 3 + 1 = -1.4567 = -ive.$

So, a root lies between 0 and 1.

The given equation can be written as

$$x = \frac{1}{3}(1 + \cos x) = g(x)$$

$$g(x) = \frac{1}{3}(1 + \cos x)$$

$$g'(x) = -\frac{1}{3}\sin x$$

$$|g'(x)| = \frac{1}{3}\sin x$$

$$|g'(0)| = 0 < 1$$

$$|g'(1)| = \frac{1}{3}\sin 1 = 0.2804 < 1$$

So the method can be applied.

Let, $x_0 = 0.6$

$$x_1 = g(x_0) = \frac{1}{3}(1 + \cos x_0) = 0.60845$$

$$x_2 = g(x_1) = \frac{1}{3}(1 + \cos x_1) = 0.60684$$

$$x_3 = g(x_2) = \frac{1}{3}(1 + \cos x_2) = 0.60715$$

$$x_4 = g(x_3) = \frac{1}{3}(1 + \cos x_3) = 0.60709$$

$$x_5 = g(x_4) = \frac{1}{3}(1 + \cos x_4) = 0.60710$$

$$x_6 = g(x_5) = \frac{1}{3}(1 + \cos x_5) = 0.60710$$

Here, $x_5 = x_6 = 0.60710$

Hence, The better approximate root is 0.60710

NEWTON'S METHOD (OR NEWTON-RAPHSON METHOD)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,2,...$$

Example:

1. Find the positive root of $x^4 - x = 10$ correct to three decimal places using Newton-Raphson method.

Solution

Given, $x^4 - x = 10$

$$f(x) = x^4 - x - 10$$
$$f(0) = 0 - 0 - 10 = -10\ (-ive)$$
$$f(1) = 1^4 - 1 - 10 = -10\ (-ive)$$
$$f(2) = 2^4 - 2 - 10 = 4\ (+ive)$$

$So,$ a root lies between 1 and 2.

Here, $|f(1)| > |f(2)|$

Therefore, the root is nearer to 2.

Let us take, $x_0 = 2$

N-R Formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,2,...$

$$f(x) = x^4 - x - 10$$

$$f'(x) = 4x^3 - 1$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put , $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{{x_0}^4 - x_0 - 10}{4{x_0}^3 - 1}$$
$$= 2 - \frac{2^4 - 2 - 10}{4(2)^3 - 1} = 2 - \frac{4}{31} = 1.87$$

$$x_1 = 1.871$$

Put , $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{{x_1}^4 - x_1 - 10}{4{x_1}^3 - 1}$$

$$= 1.871 - \frac{(1.871)^4 - 1.871 - 10}{4(1.871)^3 - 1} = 1.871 - \frac{0.383}{25.199} = 1.856$$

$$x_2 = 1.856$$

Put , $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{{x_2}^4 - x_2 - 10}{4{x_2}^3 - 1}$$

$$= 1.856 - \frac{(1.856)^4 - 1.856 - 10}{4(1.856)^3 - 1} = 1.856 - \frac{0.010}{24.574} = 1.856$$

$$x_3 = 1.856$$

Here,

$$x_2 = x_3$$

Hence, the better approximate root is 1.856

2. Find the real positive root of $3x - \cos x - 1 = 0$, by Newton's method correct to 3 decimal places.

Solution

Given, $3x - \cos x - 1 = 0$

$f(x) = 3x - \cos x - 1$

$$f(0) = 0 - 1 - 1 = -2\ (-ive)$$

$$f(1) = 3 - \cos 1 - 1 = 2 - \cos 1 = 1.459697\ (+ive)$$

$So,$ a root lies between 0 and 1.

Here, $|f(0)| > |f(1)|$

Therefore, the root is nearer to 1.

Let us take, $x_0 = 0.6$

N-R Formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,2, \ldots$

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put , $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{3x_0 - \cos x_0 - 1}{3 + \sin x_0}$$
$$= 0.6 - \frac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)} = 0.6 - \frac{(-0.025336)}{3.564642} = 0.607$$
$$x_1 = 0.607$$

Put , $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{3x_1 - \cos x_1 - 1}{3 + \sin x_1}$$
$$= 0.607 - \frac{3(0.607) - \cos(0.607) - 1}{3 + \sin(0.607)} = 0.607 - \frac{(0.000023)}{3.570495} = 0.607$$
$$x_2 = 0.607$$

Here,

$$x_1 = x_2$$

Hence, the better approximate root is 0.607

3. Find a root of $x \log_{10} x - 1.2 = 0$ by by Newton's method correct to 3 decimal places.

Solution

Given, $x \log_{10} x - 1.2 = 0$

$f(x) = x \log_{10} x - 1.2$

$$f(1) = \log_{10} 1 - 1.2 = -1.2\ (-ive)$$
$$f(2) = 2 \log_{10} 2 - 1.2 = -598\ (-ive)$$
$$f(3) = 3 \log_{10} 3 - 1.2 = 0.231\ (+ive)$$

So, a root lies between 2 and 3.

Here, $|f(2)| > |f(3)|$

Therefore, the root is nearer to 3.

Let us take, $x_0 = 2.7$

N-R Formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,2, \ldots$

$$f(x) = x \log_{10} x - 1.2$$
$$f'(x) = \left[x * \frac{1}{x} \log_{10} e\right] + \log_{10} x$$
$$= \log_{10} e + \log_{10} x$$

Put , $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0\ \log_{10} x_0 - 1.2}{\log_{10} e + \log_{10} x_0}$$
$$= 2.7 - \frac{2.7\ \log_{10} 2.7 - 1.2}{\log_{10} e + \log_{10} 2.7} = 2.7 - \frac{(-0.035)}{0.866} = 2.740$$
$$x_1 = 2.740$$

Put , $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1\ \log_{10} x_1 - 1.2}{\log_{10} e + \log_{10} x_1}$$
$$= 2.74 - \frac{2.74\ \log_{10} 2.74 - 1.2}{\log_{10} e + \log_{10} 2.74} = 2.74 - \frac{(-0.001)}{0.872} = 2.741$$
$$x_2 = 2.741$$

Put , $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2\ \log_{10} x_2 - 1.2}{\log_{10} e + \log_{10} x_2}$$
$$= 2.741 - \frac{2.741\ \log_{10} 2.741 - 1.2}{\log_{10} e + \log_{10} 2.741} = 2.741 - \frac{(0)}{0.872} = 2.741$$
$$x_3 = 2.741$$

Here,

$$x_2 = x_3$$

Hence, the better approximate root is 2.741

4. Find the iterative formula for finding the value of $\frac{1}{N}$, where N is a real number, using Newton-Raphson method. Hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Solution:

Let, $x = \frac{1}{N}$

$$ie).,N = \frac{1}{x}$$

Let, $f(x) = \frac{1}{x} - N,$

$$f'(x) = -\frac{1}{x^2}$$

N-R Formula, $x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right],\ n = 0,1,2, \ldots$

$$x_{n+1} = x_n - \left[\frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}\right]$$
$$= x_n - x_n^2 \left[\frac{1}{x_n} - N\right]$$
$$= x_n + x_n - Nx_n^2 = 2x_n - Nx_n^2$$

$x_{n+1} = x_n[2 - Nx_n]$ is the iterative formula.

To find $\frac{1}{26}, take\ N = 26$

Let $x_0 = 0.04 \left[\because\ \frac{1}{25} = 0.04\right]$

$x_{n+1} = x_n[2 - Nx_n]$

$x_1 = x_0[2 - 26x_0]$

$= (0.04)[2 - 26(0.04)] = 0.0384$

$x_1 = 0.0384$

$x_2 = x_1[2 - 26x_1]$

$= (0.0384)[2 - 26(0.0384)] = 0.0385$

$x_2 = 0.0385$

$x_3 = x_2[2 - 26x_2]$

$= (0.0385)[2 - 26(0.0385)] = 0.0385$

$x_2 = 0.0385$

Here,

$x_2 = x_3$

Hence, the value of $\frac{1}{26} = 0.0385$