Chapter-4 ***Dimensional Analysis***

INTRODUCTION

Fluid flow is influenced by several parameters like, the geometry, fluid properties and fluid velocity. In the previous chapters analytical methods used in fluid flow studies were discussed. In the study of flow of real fluids analytical methods alone are found insufficient. Experimental methods and results have contributed heavily for the development of fluid mechanics. The solution of realistic problems usually involves both anlytical and experimental studies. Experiments are used to validate analytical results as well as generalize and extend their applications. Depending either solely on analytical methods or experiments for the design of systems is found to lead to inadequate performance and high cost.

Experimental work is rather costly and time consuming, particularly when more than three parameters are involved. Hence it is necessary to plan the experiments so that most information is obtained from fewest experiments. Dimensional analysis is found to be a very useful tool in achieving this objective. The mathematical method of dimensional analysis comes to our help in this situation. The number of parameters can be reduced generally to three by grouping relevant variables to form dimensionless parameters. In addition these groups facilitate the presentation of the results of the experiments effectively and also to generalize the results so that these can be applied to similar situations.

Flow through pipes can be considered as an example. Viscosity, density, flow velocity and diameter are found to influence the flow. If the effect of each of these parameters on flow is separately studied the number of experiments will be large. Also these results cannot be generalized and its usefulness will be limited. When the number of these variables are combined to form a dimensionless group like ($u\ D\ \rho/\mu$) few experiments will be sufficient to obtain useful information. This parameter can be varied by varying one of the variables which will be the easier one to vary, for example velocity u. The results will be applicable for various combinations of these parameters and so the results can be generalized and extended to new situations. The results will be applicable also for different fluids and different diameters provided the value of the group remains the same. Example 4.1 illustrates the advantage dimensional analysis in experiment planning. The use of the results of dimensional analysis is the basis for similitude and modal studies. The topic is discussed in the next chapter.

Example *The drag force F on a stationary sphere in flow is found to depend on diameter D, velocity u, fluid density ρ and viscosity μ. Assuming that to study the influence of a parameter 10 experimental points are necessary, estimate the* ***total experimental points needed*** *to obtain complete information. Indicate how the number of experiments can be reduced.*

To obtain a curve F *vs* u, for fixed values of ρ, μ and D, experiments needed = 10.

To study the effect of ρ these 10 experiments should be repeated 10 times with 10 values of ρ the total now being 10^2.

The 10^2 experiments have to repeated 10 times each for different values of μ.

Total experiments for u, ρ and μ = 10^3.

To study the effect of variation of diameter all the experiments have to be repeated 10 times each. Hence total **experiments required = 10^4.**

These parameters can be combined to obtain two dimensionless parameters,

$$\frac{F}{\rho u^2 D^2} = f\left(\frac{\rho\, uD}{\mu}\right)$$

(The method to obtain such grouping is the main aim of this chapter)

Now only 10 experiments are needed to obtain a comprehensive information about the effect of these five parameters.

Experiments can be conducted for obtaining this information by varying the parameter $(uD\rho/\mu)$ and determining the values for $F/\rho u^2D^2$. Note : It will be almost impossible to find fluids with 10 different densities and 10 different viscosities.

METHODS OF DETERMINATION OF DIMENSIONLESS GROUPS

1. Intuitive method: This method relies on basic understanding of the phenomenon and then identifying competing quantities like types of forces or lengths etc. and obtaining ratios of similar quantities.

Some examples are: Viscous force *vs* inertia force, viscous force *vs* gravity force or roughness dimension *vs* diameter. This is a difficult exercise and considerable experience is required in this case.

2. Rayleigh method: A functional power relation is assumed between the parameters and then the values of indices are solved for to obtain the grouping. For example in the problem in example 1 one can write

$$(\pi_1, \pi_2) = F^a\, \rho^b D^c \mu^d U^e$$

The values of *a, b, c, d*, and *e* are obtained by comparing the dimensions on both sides the dimensions on the L.H.S. being zero as π terms are dimensionless. This is also tedious and considerable expertise is needed to form these groups as the number of unknowns will be more than the number of available equations. This method is also called "indicial" method.

3. Buckingham Pi theorem method: The application of this theorem provides a fairly easy method to identify dimensionless parameters (numbers). However identification of the influencing parameters is the job of an expert rather than that of a novice. This method is illustrated extensively throughout this chapter.

THE PRINCIPLE OF DIMENSIONAL HOMOGENEITY

The principle is basic for the correctness of any equation. It states "If an equation truly expresses a proper relationship between variables in a physical phenomenon, then each of the additive terms will have the same dimensions or these should be dimensionally homogeneous."

For example, if an equation of the following form expresses a relationship between variables in a process, then each of the additive term should have the same dimensions. In the expression, $A + B = C/D$, A, B and (C/D) each should have the same dimension. This principle is used in dimensional analysis to form dimensionless groups. Equations which are dimensionally homogeneous can be used without restrictions about the units adopted. Another application of this principle is the checking of the equations derived.

Note : Some empirical equations used in fluid mechanics may appear to be non homogeneous. In such cases, the numeric constants are dimensional. The value of the constants in such equations will vary with the system of units used.

BUCKINGHAM PI THEOREM

The statement of the theorem is as follows : If a relation among n parameters exists in the form

$$f(q_1, q_2, \ldots\ldots\ldots q_n) = 0$$

then the n parameters can be grouped into $n - m$ independent dimensionless ratios or π parameters, expressed in the form

$$g(\pi_1, \pi_2 \ldots\ldots\ldots \pi_{n-m}) = 0 \tag{4.3.1}$$

or

$$\pi_1 = g_1 (\pi_2, \pi_3 \ldots\ldots \pi_{n-m})$$

where m is the number of dimensions required to specify the dimensions of all the parameters, $q_1, q_2, \ldots. q_n$. It is also possible to form new dimensionless π parameters as a discrete function of the $(n - m)$ parameters. For example if there are four dimensionless parameters π_1, π_2, π_3 and π_4 it is possible to obtain π_5, π_6 etc. as

$$\pi_5 = \frac{\pi_1}{\pi_3\pi_4} \quad \text{or} \quad \pi_6 = \frac{\pi_1^{0.5}}{\pi_2^{2/3}}$$

The limitation of this exercise is that the exact functional relationship in equation 4.3.1 cannot be obtained from the analysis. The functional relationship is generally arrived at through the use of experimental results.

Determination of π π Groups

Irrespective of the method used the following steps will systematise the procedure.

Step 1. List all the parameters that influence the phenomenon concerned. This has to be very carefully done. If some parameters are left out, π terms may be formed but experiments then will indicate these as inadequate to describe the phenomenon. If unsure the parameter can be added. Later experiments will show that the π term with the doubtful parameters as useful or otherwise. Hence a careful choice of the parameters will help in solving the problem with least effort. Usually three type of parameters may be identified in fluid flow namely fluid properties, geometry and flow parameters like velocity and pressure.

Step 2. Select a set of primary dimensions, (mass, length and time), (force, length and time), (mass, length, time and temperature) are some of the sets used popularly.

Step 3. List the dimensions of all parameters in terms of the chosen set of primary dimensions. Table 8.3.1. Lists the dimensions of various parameters involved.

Table Units and Dimensions of Variables

Variable	Unit (SI)	Dimension	
		MLT θ system	FLT θ system
Mass	kg	M	FT^2/L
Length	m	L	L
Time	s	T	T
Force	N	ML/T^2	F
Temperature	deg C or K	θ	θ
Area	m^2	L^2	L^2
Volume	m^3	L^3	L^3
Volume flow rate	m^3/s	L^3/T	L^3/T
Mass flow rate	kg/s	M/T	FT/L
Velocity	m/s	L/T	L/T
Angular velocity	Rad/s	1/T	1/T
Force	N	ML/T^2	F
Pressure, stress,	N/m^2	M/LT^2	F/L^2
Bulk modulus			
Moment	Nm	ML^2/T^2	FL
Work, Energy	J, Nm	ML^2/T^2	FL
Power	W, J/s	ML^2/T^3	FL/T
Density	kg/m^3	M/L^3	FT^2/L^4
Dynamic viscosity	kg/ms, Ns/m^2	M/LT	FT/L^2
Kinematic viscosity	m^2/s	L^2/T	L^2/T
Surface tension	N/m	M/T^2	F/L
Specific heat	J/kg K	$L^2/T^2\ \theta$	$L^2/T^2\theta$
Thermal conductivity	W/mK	$ML/T^3\ \theta$	$F/T\theta$
Convective heat			
transfer coefficient	W/m^2 K	$M/T^3\ \theta$	$F/LT\theta$
Expansion coefficient	(m/m)/K	1/T	1/T

Step 4. Select from the list of parameters a set of repeating parameters equal to the number of primary dimensions. Some guidelines are necessary for the choice. (*i*) the chosen set should contain all the dimensions (*ii*) two parameters with same dimensions should not be chosen. say L, L^2, L^3, (*iii*) the dependent parameter to be determined should not be chosen.

Step 5. Set up a dimensional equation with the repeating set and one of the remaining parameters, in turn to obtain $n - m$ such equations, to determine π terms numbering $n - m$. The form of the equation is,

$$\pi_1 = q_{m+1} \cdot q_1^{\ a} \cdot q_2^{\ b} \cdot q_3^{\ c} \ldots\ldots q_m^{\ d}$$

As the LHS term is dimensionless, an equation for each dimension in terms of a, b, c, d can be obtained. The solution of these set of equations will give the values of a, b, c and d. Thus the π term will be defined.

Step 6. Check whether π terms obtained are dimensionless. This step is essential before proceeding with experiments to determine the functional relationship between the π terms.

Example .2 *The pressure drop ΔP per unit length in flow through a smooth circular pipe is found to depend on (i) the flow velocity, u (ii) diameter of the pipe, D (iii) density of the fluid* ρ, *and (iv) the dynamic viscosity* μ.

(*a*) *Using* π *theorem method, evaluate the dimensionless parameters for the flow.*

(*b*) *Using Rayleigh method (power index) evaluate the dimensionless parameters.*

Choosing the set mass, time and length as primary dimensions, the dimensions of the parameters are tabulated.

S.No.	Parameter	Unit used	Dimension
1	Pressure drop/m, ΔP	($N/m^2/m$ ($N = kgm/s^2$)	M/L^2T^2
2	Diameter, D	m	L
3	Velocity, u	m/s	L/T
4	Density, ρ	kg/m^3	M/L^3
5	Dynamic viscosity, μ	kg/ms	M/LT

There are five parameters and three dimensions. Hence two π terms can be obtained. As ΔP is the dependent variable D, ρ and μ are chosen as repeating variables.

Let $\pi_1 = \Delta P\, D^a \rho^b u^c$, Substituting dimensions,

$$M^0L^0T^0 = \frac{M}{L^2T^2} L^a \frac{M^b}{L^{3b}} \frac{L^c}{T^c}$$

Using the principle of dimensional homogeneity, and in turn comparing indices of mass, length and time.

$$1 + b = 0 \quad \therefore \quad b = -1, \quad -2 + a - 3b + c = 0 \ \therefore \quad a + c = -1$$
$$-2 - c = 0 \quad \therefore \quad c = -2, \quad \text{Hence } a = 1.$$

Substituting the value of indices we obtain

$\therefore \qquad \pi_1 = \Delta PD/\rho u^2;$

This represents the ratio of pressure force and inertia force.

Check the dimension :

$$\frac{M}{L^2T^2}\, L\, \frac{L^3}{M}\, \frac{T^2}{L^2} = M^0L^0T^0$$

Let $\pi_2 = \mu\, D^a\, \rho^b u^c$, substituting dimensions and considering the indices of *M, L* and *T,*

$$M^0L^0T^0 = \frac{M}{LT}\, L^a\, \frac{M^b}{L^{3b}}\, \frac{L^c}{T^c}$$

$$1 + b = 0 \quad \text{or} \quad b = -1, \quad -1 + a - 3b + c = 0, \quad a + c = -2, -1 - c = 0, \quad c = -1 \quad a = -1$$

Substituting the value of indices,

$\therefore \qquad \pi_2 = \mu/u\rho D$

check, $$\frac{M}{LT}\, \frac{T}{L}\, \frac{L^3}{M}\, \frac{1}{L} = M^0L^0T^0$$

This term may be recognised as inverse of Reynolds number. So π_2 can be modified as $\pi_2 = \rho uD/\mu$ also $\pi_2 = (uD/v)$. The significance of this π term is that it is the ratio of inertia force to viscous force. In case *D, u* and μ had been choosen as the repeating, variables, $\pi_1 = \Delta PD^2/u\,\mu$ and $\pi_2 = \rho Du/\mu$. The parameter π_1/π_2 will give the dimensionless term. $\Delta P\; D/\rho u^2$. In this case π_1 represents the ratio pressure force/viscous force. This flow phenomenon is influenced by the three forces namely pressure force, viscous force and inertia force.

Rayleigh method: (Also called method of Indices). The following functional relationship is formed first. There can be two p terms as there are five variables and three dimensions.

$$DP^aD^b r^c m^d u^e = (p_1\ p_2), \text{ Substituting dimensions,}$$

$$\frac{M^a}{L^{2a}\,T^{2a}}\, L^b\, \frac{M^c}{L^{3c}}\, \frac{M^d}{L^dT^d}\, \frac{L^e}{T^e} = L^0\, M^0\, T^0$$

Considering indices of *M, L* and *T*, three equations are obtained as below

$$a + c + d = 0, -2a + b - 3c - d + e = 0, -2a + d - e = 0$$

There are five unknowns and three equations. Hence some assumptions are necessary based on the nature of the phenomenon. As D*P*, the dependent variable can be considered to appear only once. We can assume $a = 1$. Similarly, studying the forces, m appears only in the viscous force. So we can assume $d = 1$. Solving $a = 1$, $d = 1$, $b = 0$, $c = -2$, $e = -3$, $(p_1\ p_2) = DPm/r^2\, u^3$. Multiply and divide by *D*, then $p_1 = DPD/ru^2$ and $p_2 = m/ruD$. Same as was obtained by p theorem method. This method requires more expertise and understanding of the basics of the phenomenon.

Example .3 *The pressure drop ΔP in flow of incompressible fluid through rough pipes is found to depend on the length l, average velocity u, fluid density,* ρ, dynamic viscosity μ, *diameter D and average roughness height e.* ***Determine the dimensionless groups*** *to correlate the flow parameters.*

The variables with units and dimensions are listed below.

S.No.	Variable	Unit	Dimension
1	ΔP	N/m^2	M/LT^2
2	l	L	L
3	u	m/s	L/T
4	ρ	kg/m^3	M/L^3
5	μ	kg/ms	M/LT
6	D	L	L
7	e	L	L

There are seven parameters and three dimensions. So four π terms can be identified. Selecting u, D and ρ as repeating variables, (as these sets are separate equations, no problem will arise in using indices a, b and c in all cases).

Let $\pi_1 = \Delta P\, u^a D^b \rho^c,\ \pi_2 = L\, u^a D^b\, \rho^c,\ \pi_3 = \mu\, u^a\, D^b\, \rho^c,\ \pi_4 = e\, u^a D^b \rho^c$

Consider π_1, $M^0L^0T^0 = \dfrac{M}{LT^2}\dfrac{L^a}{T^a} L^b \dfrac{M^c}{L^{3c}}$

Equating the indices of M, L and T,

$$1 + c = 0,\ c = -1,\ -1 + a + b - 3c = 0,\ -2 - a = 0,\quad a = -2,\ b = 0.$$

Substituting the value of indices we get

$\therefore$ $\boldsymbol{\pi_1 = \Delta P/\rho u^2}$

Consider π_2, $M^0L^0T^0 = L\dfrac{L^a}{T^a} L^b \dfrac{M^c}{L^{3c}}$

Equating indices of M, L and T, $c = 0$, $1 + a + b - 3c = 0$, $a = 0$, $\therefore$ $b = -1$, $\therefore$ $\boldsymbol{\pi_2 = L/D}$

Consider π_3 $M^0L^0T^0 = \dfrac{M}{LT}\dfrac{L^a}{T^a} L^b \dfrac{M^c}{L^{3c}}$

Comparing the indices of μm, L and T,

gives $1 + c = 0$ or $c = -1,\ -1 + a + b - 3c = 0,\ -1 - a = 0$ or $a = -1$, $\therefore$ $b = -1$

$\therefore$ $\boldsymbol{\pi_3 = \mu/\rho D u}$ **or** $\boldsymbol{\rho u D/\mu}$

Consider π_4, $M^0L^0T^0 = L\dfrac{L^a}{T^a} L^b \dfrac{M^c}{L^{3c}}$

This gives, $c = 0$, $1 + a + b - 3c = 0$, $-a = 0$, $b = -1$ $\therefore$ $\boldsymbol{\pi_4 = e/D}$

These π terms may be checked for dimensionless nature.

The relationship can be expressed as $\dfrac{\Delta P}{\rho u^2} = f\left[\dfrac{L}{D}, \dfrac{e}{D}, \dfrac{\rho u D}{\mu}\right]$

IMPORTANT DIMENSIONLESS PARAMETERS

Some of the important dimensionless groups used in fluid mechanics are listed in Table 4.4.1. indicating significance and area of application of each.

Table Important Dimensionless Parameters

Name	Description	Significance	Applications
Reynolds Number, Re	$\rho uD/\mu$ or uD/v	Inertia force/ Viscous force	All types of fluid dynamics problems
Froude Number Fr	$u/(gl)^{0.5}$ or u^2/gl	Inertia force/ Gravity force	Flow with free surface (open channel and ships)
Euler Number Eu	$P/\rho u^2$	Pressure force/ Inertia force	Flow driven by pressure
Cauchy Number Ca	$\rho u^2/E_v$ (E_v– bulk modulous)	Inertia force/ Compressibility force	compressible flow
Mach Number M	u/c, c–Velocity of sound	Inertia force/ Compressibility force	Compressible flow
Strouhal Number St	$\omega l/u$, ω–Frequency of oscillation	Local inertia Force/ Convective inertia force	Unsteady flow with frequency of oscillation
Weber Number We	$\rho u^2 l/\sigma$, σ = Surface tension	Inertia force/ Surface tension force	Problems influenced by surface tension free surface flow
Lift coefficient C_L	$L/(1/2\ \rho A u^2)$ L = lift force	Lift force/ Dynamic force	Aerodynamics

CORRELATION OF EXPERIMENTAL DATA

Dimensional analysis can only lead to the identification of relevant dimensionless groups. The exact functional relations between them can be established only by experiments. The degree of difficulty involved in experimentation will depend on the number of π terms.

Problems with One Pi Term

In this case a direct functional relationship will be obtained but a constant c has to be determined by experiments. The relationship will be of the form $\pi_1 = c$. This is illustrated in example 8.4.

Example .4 *The drag force acting on a spherical particle of diameter D falling slowly through a viscous fluid at velocity u is found to be influenced by the diameter D, velocity of fall u, and the viscosity* μ. *Using the method of dimensional analysis obtain a relationship between the variables.*

The parameters are listed below using *M, L, T* dimension set.

S.No.	Parameter	Symbol	Unit	Dimension
1	Drag Force	F	N or kgm/s^2	ML/T^2
2	Diameter	D	m	L
3	Velocity	u	m/s	L/T
4	Viscosity	μ	kg/ms	M/LT

There are four parameters and three dimensions. Hence only one π term will result.

$\pi_1 = F\,D^a\,u^b\,\mu^c$, Substituting dimensions,

$$M^0L^0T^0 = \frac{ML}{T^2}\,L^a\,\frac{L^b}{T^b}\,\frac{M^c}{L^cT^c}\,,$$ Equating indices of *M, L* and *T*

$$0 = 1 + c,\ c = -1,\ 1 + a + b - c = 0,\ 2 + b + c = 0,\ b = -1,\ c = -1$$

$$\boldsymbol{\pi_1 = F/uD\mu \qquad \therefore\ F/uD\mu = \text{constant} = c}$$

or $F = cuD\mu$ or drag force varies directly with velocity, diameter and viscosity. A single test will provide the value of the constant. However, to obtain a reliable value for c, the experiments may have to be repeated changing the values of the parameters.

In this case an approximate solution was obtained theoretically for c as 3π. Hence drag force F in free fall is given by $F = 3\pi\mu uD$. This can be established by experiments.

This relation is known as Stokes law valid for small values of Reynolds Number (Re << 1). This can be used to study the settling of dust in still air. Inclusion of additional variable, namely density will lead to another π term.

Problems with Two Pi Terms

In example 4.2 two π terms were identified. If the dimensional analysis is valid then a single universal relationship can be obtained. Experiments should be conducted by varying one of the group say π_1 and from the measurement the values of the other group π_2 is calculated. A suitable graph (or a computer program) can lead to the functional relationship between the π terms. Linear semilog or log/log plots may have to be used to obtain such a relationship. The valid range should be between the two extreme values used in the experiment. Extrapolation may lead to erroneous conclusions. This is illusration by example 4.5.

Example .5 *In order to determine the pressure drop in pipe flow per m length an experiment was conducted using flow of water at 20°C through a 20 mm smooth pipe of length 5 m. The variation of pressure drop observed with variation of velocity is tabulated below.The density of water = 1000 kg/m^3. Viscosity = 1.006 × 10^{-3} kg/ms.*

Velocity, m/s	0.3	0.6	0.9	1.5	2.0	3	5
Pressure drop, N	404	1361	2766	6763	11189	22748	55614

Determine the functional relationship between the dimensionless parameters ($D\,\Delta P/\rho u^2$) and ($\rho uD/\mu$).

Using the data the two π parameters together with log values are calculated and tabulated below.

u	0.3	0.6	0.9	1.5	2.0	3	5
$D\Delta P/\rho u^2$	0.01798	0.01512	0.01366	0.01202	0.01119	0.01011	0.00890
$\rho uD/\mu$	5964	11928	17894	29821	39761	59642	99400
logRe	3.78	4.08	4.25	4.48	4.6	4.78	4.997
log($D\Delta P/\rho u$)	– 1.745	– 1.821	– 1.865	– 1.92	– 1.951	– 1.995	– 2.051

A plot of the data is shown in Fig. Ex. 4.5 (*a*). The correlation appears to be good. Scatter may indicate either experimental error or omission of an influencing parameter. As the direct plot is a curve., fitting an equation can not be done from the graph. A log log plot results in a straight line, as shown in the Fig. 8.5 (*b*). To fit an equation the following procedure is used.

The slope is obtained by taking the last values:

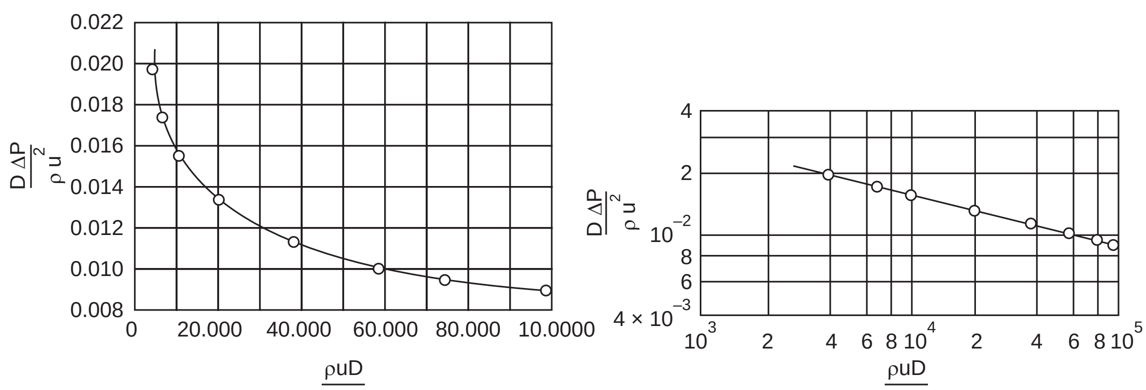
$$= \{-2.051 - (-1.745)\}/(4.997 - 3.78) = -0.2508$$

When extrapolating we can write, the slope using the same – 2.051 – (x)/(5 – 0) = – 0.2508

This gives $x = -0.797$.

This corresponds to the value of 0.16. Hence we can write,

$$\frac{D\Delta P}{\rho u^2} = 0.16\left(\frac{PuD}{\mu}\right)^{-0.2508} = 0.16 \times \mathrm{Re}^{-0.2508}$$



(a) (b)

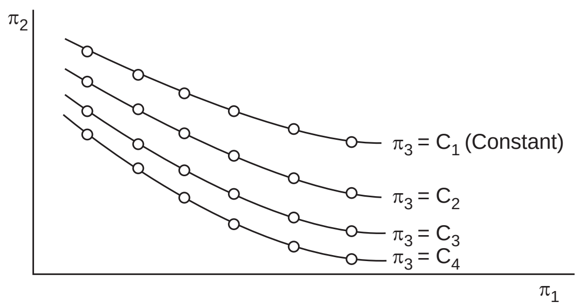
Problems with Three Dimensionless Parameters

In this case experiments should be conducted for different constant values of π_3, varying π_1 and calculating the corresponding values of π_2. Such a set of experiments will result in curves of the form shown in Fig. 4.5.3.

These curves can also be converted to show the variation of π_1 with π_3 at constant values of π_2 by taking sections at various values of π_2. By suitable mathematical techniques correlation of the form below can be obtained.

$$\pi_2 = c\pi_1^{n_1}\ \pi_2^{n_2}$$

When there are more than three π terms, two of these should be combined and the numbers reduced to three. The procedure as described above can then be used to obtain the functional relationship.



SOLVED PROBLEMS

Problem .1 *The pressure drop ΔP in flow through pipes per unit length is found to depend on the average velocity* μ, *diameter D, density of the fluid* ρ, *and viscosity* μ. *Using FLT set of dimensions evaluate the dimensionless parameters correlating this phenomenon.*

The dimensions of the influencing parameters are tabulated below choosing FLT set.

S.No.	Variables	Unit	Dimensions
1	Pressure drop per unit length, $\Delta P/l$	$(N/m^2)/m$	F/L^3
2	Diameter, D	m	L
3	Velocity, u	m/s	L/T
4	Density, ρ	kg/m^3	FT^2/L^4
5	Viscosity, μ	Ns/m^2	FT/L^2

As there are five variables and three dimensions, two π terms can be obtained.

Using D, u and ρ as repeating parameters,

Let $$\pi_1 = \Delta P d^a u^b \rho^c \quad \text{or} \quad F^0 L^0 T^0 = \frac{F}{L^3} L^a \frac{L^b}{T^b} \frac{F^c\, T^{2c}}{L^{4c}}$$

Comparing the indices of M, L and T solving for a, b and c,

$$1 + c = 0, -3 + a + b - 4c = 0, -b + 2c = 0$$

$$\therefore \quad c = -1, b = -2, a = 1$$

Substituting the value of indices

$$\therefore \quad \boldsymbol{\pi_1 = D\Delta P/\rho u^2}$$

Let, $$\pi_2 = \mu D^a\, u^b\, \rho^c, \quad \text{or} \quad F^0\, L^0\, T^0 = \frac{F}{L^2} L^a \frac{L^b}{T^b} \frac{F^c\, T^{2c}}{L^{4c}}$$

Comparing the value of indices for M, L and T

$$\therefore \quad 1 + c = 0, -2 + a + b - 4c = 0, 1 - b + 2c = 0$$

Solving, $a = -1$, $b = -1$, $c = -1$ substituting the values of a, b, c, d

$$\therefore \quad \boldsymbol{\pi_2 = \mu/\rho u D} \quad \text{or} \quad \boldsymbol{\rho u D/\mu}$$

$$\therefore \quad \frac{D\Delta P}{\rho u^2} = f\left[\frac{\rho u D}{\mu}\right]$$

The result is the same as in example 4.2. The dimension set choosen should not affect the final correlation.

Problem.2 *The drag force on a smooth sphere is found to be affected by the velocity of flow, u, the dimaeter D of the sphere and the fluid properties density* ρ *and viscosity* μ. *Using dimensional analysis obtain the dimensionless groups to correlate the parameters.*

The dimensions of the influencing variables are listed below, using M, L, T set.

S.No.	Variables	Unit	Dimensions
1	Drag force, F	N, (kgm/s^2)	ML/T^2
2	Diameter, D	m	L
3	Velocity, u	m/s	L/T
4	Density, ρ	kg/m^3	M/L^3
5	Viscosity, μ	kg/ms	M/LT

There are five variables and three dimensions. So two π terms can be obtained. Choosing D, u and ρ as repeating variables,

Let $$\pi_1 = F\, D^a\, u^b\, \rho^c, \quad \text{or} \quad M^0 L^0 T^0 = \frac{ML}{T^2} L^a \frac{L^b}{T^b} \frac{M^c}{L^{3c}}$$

Comparing the values of indices for M, L and T

$$1 + c = 1, \quad \therefore \quad c = -1, 1 + a + b - 3c = 0, -2 - b = 0$$

$$\therefore \quad b = -2, a = -2$$

Substituting the values of a, b, c

$$\therefore \quad \boldsymbol{\pi_1 = F/\rho u^2 D^2}$$

Let $\quad \pi_2 = \mu D^a u^b \rho^c \quad$ or $\quad M^0L^0T^0 = \dfrac{M}{LT} L^a \dfrac{L^b}{T^b} \dfrac{M^c}{L^{3c}}$

Comparing the values of indices of M, L and T

$$\therefore \quad 1 + c = 0, -1 + a + b - 3c = 0, -1 - b = 0 \quad \therefore \quad c = -1, b = -1, a = -1$$

Susbtituting the values of a, b, c.

$$\therefore \quad \boldsymbol{\pi_2 = \mu/\rho u D \quad \text{or} \quad \rho u D/\mu}$$

$\therefore \quad \dfrac{F}{\rho u^2 D^2} = f\left[\dfrac{\rho\, uD}{\mu}\right]$; Check for dimensions of π_1 and π_2.

$$\pi_1 = \frac{ML}{T^2}\frac{L^3}{M}\frac{T^2}{L^2}\frac{1}{L^2} = M^0L^0T^0 \quad \text{or} \quad \pi_2 = \frac{M}{L^3}\frac{L}{T} L \frac{LT}{M} = M^0L^0T^0$$

Note: the significance of the π term. $F/\rho u^2D^2 \to F/\rho u\, Du \to F/mu \to$ Drag force/inertia force.

Problem 3. *The thrust force, F generated by a propeller is found to depend on the folllowing parameters: diameter D, forward velocity u, density* ρ, *viscosity* μ *and rotational speed N. Determine the dimensionless parameters to correlate the phenomenon.*

The influencing parameters with dimensions are listed below using *MLT* set.

S.No.	Parameters	Unit	Dimensions
1	Thrust force, F	N	ML/T^2
2	Diameter, D	m	L
3	Forward velocity, u	m/s	L/T
4	Density, ρ	kg/m^3	M/L^3
5	Viscosity, μ	kg/ms	M/LT
6	Rotational speed, N	1/s	$1/T$

There are 6 variables and three dimensions. So three π terms can be obtained. Choosing D, u and ρ as repeating variables,

Let $\quad \pi_1 = F u^a D^b \rho^c, \quad$ or $\quad M^0L^0T^0 = \dfrac{ML}{T^2}\dfrac{L^a}{T^a} L^b \dfrac{M^c}{L^{3c}}$

Comparing indices of M, L and T

$\therefore$ $1 + c = 0, 1 + a + b - 3c = 0, -2 - a = 0$

$\therefore$ $a = -2, b = -2, c = -1$

Substituting the values of $a, b,$ and c

$\therefore$ $\boldsymbol{\pi_1 = F/u^2D^2\,\rho,}$ (Thrust force/Inertia force)

Let $\pi_2 = \mu\, u^a D^b \rho^c$ or $M^0L^0T^0 = \dfrac{M}{LT}\,\dfrac{L^a}{T^a}\,L^b\,\dfrac{M^c}{L^{3c}}$

Comparing the indices M, L and T

$\therefore$ $1 + c = 0, -1 + a + b - 3c = 0, -1 - a = 0$

$\therefore$ $a = -1, b = -1, c = -1$

Substituting the values of a, b and c

$\therefore$ $\boldsymbol{\pi_2 = \mu/\rho u D}$ **or** $\boldsymbol{\rho u D/\mu}$ (Inertia force/Viscous force)

Let $\pi_3 = N\, u^a\, D^b\, \rho^c,$ or $M^0L^0T^0 = \dfrac{1}{T}\,\dfrac{L^a}{T^a}\,L^b\,\dfrac{M^c}{L^{3c}}$

Comparing the indices of M, L and T

$c = 0, a + b - 3c = 0, -1 - a = 0, \quad \therefore \quad a = -1, b = 1$

Susbtituting the values of a, b and c

$\therefore$ $\boldsymbol{\pi_3 = ND/u}$ (Rotational speed/Forward speed)

$$\therefore \quad F/u^2D^2\,\rho = f\left[\frac{uD\rho}{\mu}, \frac{ND}{u}\right]$$

Problem 4. *At higher speeds where compressibility effects are to be taken into account the performance of a propeller in terms of force exerted is influenced by the diameter, forward speed, rotational speed, density, viscosity and bulk modulus of the fluid. Evaluate the dimensionless parameters for the system.*

The influencing parameters and dimensions are tabulated below, using M, L, T set.

S.No.	Parameters	Unit	Dimensions
1	Force, F	N/m^2	M/LT^2
2	Diameter, D	m	L
3	Forward velocity, u	m/s	L/T
4	Rotational speed, N	l/s	$1/T$
5	Density, ρ	kg/m^3	M/L^3
6	Viscosity, μ	kg/ms	M/LT
7	Bulk Modulus, E	$(m^3/m^3)N/m^2$	M/LT^2

There are seven variables and three dimensions, So four π terms are possible. Selecting D, u and ρ as repeating parameters,

Let $$\pi_1 = F\,\rho^a u^b d^c, \quad \text{or} \quad M^0L^0T^0 = \frac{M}{LT^2}\frac{M^a}{L^{3a}}\frac{L^b}{T^b}L^c$$

The general procedure is to compare the indices of *M, L* and T on both sides and from equations.

$$1 + a = 0, -1 - 3a + b + c = 0, -2 - b = 0$$

$\therefore$ $$c = 0, b = -2, a = -1$$

$\therefore$ $$\boldsymbol{\pi_1 = F/\rho u^2} \rightarrow \text{(force exerted/inertia force)/m}^2$$

Let $$\pi_2 = N\rho^a u^b D^c, \quad \text{or} \quad M^0L^0T^0 = \frac{1}{T}\frac{M^a}{L^{3a}}\frac{L^b}{T^b}L^c$$

$\therefore$ $$a = 0, -3a + b + c = 0, -1 - b = 0$$

$\therefore$ $$a = 0, b = -1, c = 1.$$

$$\boldsymbol{\pi_2 = ND/u} \text{ (or rotational speed/forward speed)}$$

Let $$\pi_3 = \mu\rho^a u^b D^c, \quad \text{or} \quad M^0L^0T^0 = \frac{M}{LT}\frac{M^a}{L^{3a}}\frac{L^b}{T^b}L^c$$

$\therefore$ $$1 + a = 0, -1 - 3a + b + c = 0, -1 - b = 0$$

$\therefore$ $$a = -1, b = -2, c = -1,$$

$\therefore$ $$\boldsymbol{\pi_3 = \mu/\rho uD \quad \text{or} \quad \rho uD/\mu} \quad \text{(Reynolds number)}$$

Let $$\pi_4 = E\rho^a u^b D^c, \quad \text{or} \quad M^0L^0T^0 = \frac{M}{LT^2}\frac{M^a}{L^{3a}}\frac{L^b}{T^b}L^c$$

$\therefore$ $$1 + a = 0, -1 - 3a + b + c = 0, -2 - b = 0$$

$\therefore$ $$a = -1, b = -2, c = 0$$

$$\boldsymbol{\pi_4 = E/\rho u^2} \quad \text{(Compressibility force/inertia force)}$$

$\therefore$ $$\frac{F}{\rho u^2} = f\left[\frac{ND}{u}, \frac{\rho uD}{\mu}, \frac{E}{\rho u^2}\right]$$

Problem 5. *Using dimensional analysis, obtain a correlation for the frictional torque due to rotation of a disc in a viscous fluid. The parameters influencing the torque can be identified as the diameter, rotational speed, viscosity and density of the fluid.*

The influencing parameters with dimensions are listed below, using *M, L, T* set.

S.No.	Parameters	Unit	Dimensions
1	Torque, τ	Nm	ML^2/T^2
2	Diameter, D	m	L
3	Rotational speed, N	l/s	$1/T$
4	Density, ρ	kg/m^3	M/L^3
5	Viscosity, μ	kg/ms	M/LT

There are five variables and three dimensions. So two π parameters can be identified.

Considering D, N and ρ as repeating variables.

Let $\pi_1 = \tau\, D^a N^b \rho^c$ or $M^0L^0T^0 = \dfrac{ML^2}{T^2} L^a \dfrac{1}{T^b} \dfrac{M^c}{L^{3c}}$

$\therefore$ $1 + c = 0,\ 2 + a - 3c = 0,\ \ -2 - b = 0$ $\therefore$ $c = -1,\ b = -2,\ a = -5,$

$\therefore$ $\boldsymbol{\pi_1 = \tau/\rho N^2 D^5}$

Let $\pi_2 = \mu\, D^a N^b \rho^c$ or $M^0L^0T^0 = \dfrac{M}{LT} L^a \dfrac{1}{T^b} \dfrac{M^c}{L^{3c}}$

$\therefore$ $1 + c = 0,\ -1 + a - 3c = 0,\ -1 - b = 0$ $\therefore$ $c = -1,\ b = -1,\ a = -2$

$\therefore$ $\boldsymbol{\pi_2 = \mu/\rho D^2 N}$**,** (Another form of Reynolds number, as DN $\to u$)

$\therefore$ $\dfrac{\tau}{\rho N^2 D^5} = f\left[\dfrac{\mu}{\rho D^2 N}\right]$ Check for the dimensions of π_1 and π_2

Note: Rotational speed can also be expressed as angular velocity, ω. In that case N will be replaced by ω as the dimension of both these variables is $1/T$.

Problem 6. *A rectangular plate of height, a and width, b is held perpendicular to the flow of a fluid. The drag force on the plate is influenced by the dimensions a and b, the velocity u, and the fluid properties, density* ρ *and viscosity* μ. *Obtain a correlation for the drag force in terms of dimensionless parameters.*

The parameters with dimensions are listed adopting M, L, T set of dimensions.

S.No.	Parameters	Unit	Dimensions
1	Drag force, F	N	ML/T^2
2	Width, b	m	L
3	Height, a	m	L
4	Velocity, u	m/s	L/T
5	density, ρ	kg/m^3	M/L^3
6	Viscosity, μ	kg/ms	M/LT

There are 6 parameters and three dimensions. Hence three π terms can be obtained. Selecting b, u and ρ as repeating variables.

Let $\pi_1 = F\, b^a u^b \rho^c$ or $M^0L^0T^0 = \dfrac{ML}{T^2} L^a \dfrac{L^b}{T^b} \dfrac{M^c}{L^{3c}}$

$\therefore$ $1 + c = 0,\ 1 + a + b - 3c = 0,\ -2 - b = 0$

$\therefore$ $c = -1,\ b = -2,\ a = -2$

$\therefore$ $\boldsymbol{\pi_1 = F/\rho u^2 b^2}$

Let $\pi_2 = a\, b^a u^b \rho^c$ or $M^0L^0T^0 = LL^a \dfrac{L^b}{T^b} \dfrac{M^c}{L^{3c}}$

$c = 0,\ 1 + a + b - 3c = 0,\ -b = 0,$

$\therefore$ $a = -1$ $\therefore$ $\boldsymbol{\pi_2 = a/b}$

Let $\boldsymbol{\pi_3 = \mu b^a u^b \rho^c}$ or $M^0L^0T^0 = \dfrac{M}{LT} L^a \dfrac{L^b}{T^b} \dfrac{M^c}{L^{3c}}$

$\therefore$ $1 + c = 0,\ -1 + a + b - 3c = 0,\ -1 - b = 0.$

$\therefore$ $b = -1,\ c = -1,\ b = -1$

$\therefore$ $\boldsymbol{\pi_3 = \mu/\rho u b}$ **or** $\boldsymbol{\pi_3 = \rho u b/\mu}$

$$\therefore \quad \frac{F}{\rho u^2 b^2} = f\left[\frac{a}{b}, \frac{\rho\, ub}{\mu}\right]$$

π_3 is Reynolds number based on length b. π_1 is (drag force/unit area)/inertia force.

Problem 7. *In film lubricated journal bearings, the frictional torque is found to depend on the speed of rotation, viscosity of the oil, the load on the projected area and the diameter. Evaluate dimensionless parameters for application to such bearings in general.*

The variables with dimensions are listed below, adopting *MLT* set.

S.No.	Variable	Unit	Dimensions
1	Frictional Torque, τ	Nm	ML^2/T^2
2	Speed, N	1/s	$1/L$
3	Load per unit area, P	N/m^2	M/LT^2
4	Diameter, D	m	L
5	Viscosity, μ	kg/ms	M/LT

There are five parameters and three dimensions. Hence two π parameters can be found. Considering N, D and μ as repeating variables,

Let $\pi_1 = \tau N^a D^b \mu^c$ or $M^0L^0T^0 = \dfrac{ML^2}{T^2} \dfrac{1}{T^a} L^b \dfrac{M^c}{L^cT^c}$

$\therefore$ $1 + c = 0,\ 2 + b - c = 0,\ -2 - a - c = 0$ $\therefore$ $c = -1,\ a = -1,\ b = -3$

$\therefore$ $\boldsymbol{\pi_1 = \tau/N\mu D^3}$ Also $\pi = \tau/\mu u D$ (τ–Torque)

Let $\pi_2 = PN^aD^b\mu^c$ or $M^0L^0T^0 = \dfrac{M}{LT^2} \dfrac{1}{T^a} L^b \dfrac{M^c}{L^cT^c}$

$\therefore$ $1 + c = 0,\ -1 + b - c = 0,\ -2,\ -a - c = 0$

$\therefore$ $c = -1,\ a = -1,\ b = 0$

$\therefore$ $\boldsymbol{\pi_2 = P/N\mu,}$ $\therefore$ $\dfrac{\tau}{N\mu D^3} = f\left[\dfrac{P}{N\mu}\right]$

Note : $P/N\mu$ is also Reynolds number, try to verify.

Problem 8. *Obtain a relation using dimensional analysis, for the resistance to uniform motion of a partially submerged body (like a ship) in a viscous compressible fluid.*

The resistance can be considered to be influenced by skin friction forces, buoyant forces and compressibility of the fluid.

The variables identified as affecting the situation are listed below using *MLT* set.

S.No.	Variable	Unit	Dimensions
1	Resistance to motion, R	N/m^2	M/LT^2
2	Forward velocity, u	m/s	L/T
3	Length of the body, l	m	L
4	Density of the fluid, ρ	kg/ms	M/L^3
5	Viscosity, of the fluid, μ	kg/ms	M/LT
6	Gravity, g	m/s^2	L/T^2
7	Bulk modulus, E_v	N/m^2	M/LT^2

There are seven parameters and three dimensions. So four π, terms are possible. Considering velocity, density and length as repeating variables.

Let $\quad \pi_1 = Ru^a\rho^b l^c \quad$ or $\quad M^0L^0T^0 = \dfrac{M}{LT^2}\dfrac{L^a}{T^a}\dfrac{M^b}{L^{3b}}L^c$

$\therefore \quad 1 + b = 0, -1 + a - 3b + c = 0, \quad -2 - a = 0$

$\therefore \quad a = -2, b = -1$ and $c = 0$

$\therefore \quad$ $\boldsymbol{\pi_1 = R/\rho u^2}$, Euler number.

Let $\quad \pi_2 = \mu\, u^a\, \rho^b\, l^c \quad$ or $\quad M^0L^0T^0 = \dfrac{M}{LT}\dfrac{L^a}{T^a}\dfrac{M^b}{L^{3b}}L^c$

$\therefore \quad 1 + b = 0, -1 + a - 3b + c = 0, -2 - a = 0$

$\therefore \quad a = -1, b = -1 \quad$ and $\quad c = -1$

$\therefore \quad$ $\boldsymbol{\pi_2 = \mu/u\rho l}$.

Let $\quad \pi_3 = gu^a\rho^b l^c \quad$ or $\quad M^0L^0T^0 = \dfrac{L}{T^2}\dfrac{L^a}{T^a}\dfrac{M^b}{L^{3b}}L^c$

$\therefore \quad b = 0, 1 + a - 3b + c = 0, -2 - a = 0$

$\therefore \quad a = -2, b = 0 \quad$ and $\quad c = 1$

$\therefore \quad$ $\boldsymbol{\pi_3 = gl/u^2}$ $\rightarrow$ can also be expressed as $u/(gL)^{0.5}$ (Froude number.)

Let $\quad \pi_4 = E_v u^a\rho^b l^c \quad$ or $\quad M^0L^0T^0 = \dfrac{M}{LT^2}\dfrac{L^a}{T^a}\dfrac{M^b}{L^{3b}}L^c$

$\therefore \quad 1 + b = 0, -1 + a - 3b + c = 0, -2 - a = 0$

$\therefore \quad a = -2, b = -1$ and $c = 0$

$\therefore \quad$ $\boldsymbol{\pi_4 = E_v/\rho u^2}$

$$\frac{R}{\rho u^2} = f\left[\frac{\mu}{ul\rho}, \frac{gl}{u^2}, \frac{E_v}{\rho u^2}\right] \quad \text{or}$$

Euler number = f (Reynolds number, Froude number and Mach number)

In the case of incompressible flow, this will reduce to

$$\frac{R}{\rho u^2} = f\left[\frac{ul\rho}{\mu}, \frac{u}{(gl)^{0.5}}\right] = f(\text{Re, Fr})$$

Problem 9. *The velocity of propagation of pressure wave, c through a fluid is assumed to depend on the fluid density ρ and bulk modulus of the fluid E_v. Using dimensional analysis obtain an expression for c in terms of ρ and E_v.*

This is a case were there will be a direct relationship between the variables or one π term.

Note: The definition of the bulk modulus is $dp/(dv/v)$, the dimension being that of pressure, M/LT^2, Writing $c = f(\rho, E_v)$

Let, $\pi_1 = c\,\rho^a E_v^{\,b}$ or $M^0L^0T^0 = \dfrac{L}{T}\dfrac{M^a}{L^a}\dfrac{M^b}{L^b\,T^{2b}}$

∴ $a + b = 0,\ 1 - 3a - b = 0,\ -1 - 2b = 0,\quad \therefore\ b = -0.5,\ a = 0.5$

∴ $\boldsymbol{\pi_1 = c(\rho/E_v)^{0.5}}$, or $c = \text{const} \times (E_v/\rho)^{0.5}$

Problem 10. *Obtain a correlation for the coefficient of discharge through a small orifice, using the method of dimensional analysis.*

The following list of parameters can be identified as affecting the coefficient of discharge

S.No.	Parameters	Unit	Dimensions
1	Diameter, D	m	L
2	Head, H	m	L
3	Gravity, g	m/s^2	L/T^2
4	Density of the fluid, ρ	kg/m^3	M/L^3
5	Roughness height, k	m	L
6	Surface tension, σ	N/m	M/T^2
7	Viscosity, μ	kg/ms^2	M/LT

There are seven variables and three dimensions. So four π terms can be identified. Considering ρ, g and H as repeating variables

Let $\pi_1 = D\,\rho^a g^b H^c$ or $M^0L^0T^0 = L\,\dfrac{M^a}{L^{3a}}\dfrac{L^b}{T^{2b}}\,L^c$

∴ $a = 0,\ 1 - 3a + b + c = 0,\ -2b = 0,$

∴ $c = -1 \quad \therefore\ \boldsymbol{\pi_1 = D/H}$ **or H/D**

Let $\pi_2 = k\rho^a g^b H^c$ or $M^0L^0T^0 = L\,\dfrac{M^a}{L^{3a}}\,\dfrac{L^b}{T^{2b}}\,L^c$

$\therefore$ $a = 0,\ 1 - 3a + b + c = 0,\ -2b = 0,$

$\therefore$ $a = 0,\ b = 0,\ c = -1$ $\therefore$ $\boldsymbol{\pi_2 = k/H}$

Let $\pi_3 = \sigma\rho^a g^b H^c$ or $M^0L^0T^0 = \dfrac{M}{T^2}\,\dfrac{M^a}{L^{3a}}\,\dfrac{L^b}{T^{2b}}\,L^c$

$\therefore$ $a + 1 = 0,\ -3a + b + c = 0,\ -2 - 2b = 0,$

$\therefore$ $a = -1,\ b = -1,\ c = -2$

$\therefore$ $\boldsymbol{\pi_3 = \sigma/\rho g H^2}$

Let $\pi_4 = \mu\rho^a g^b H^c$ or $M^0L^0T^0 = \dfrac{M}{LT}\,\dfrac{M^a}{L^{3a}}\,\dfrac{L^b}{T^{2b}}\,L_c$

$\therefore$ $a + 1 = 0,\ -1 - 3a + b + c = 0,\ -1 - 2b = 0,$

$\therefore$ $a = -1,\ b = -1/2,\ c = -1.5.$

$\therefore$ $\boldsymbol{\pi_4 = \mu/(\rho H\sqrt{gH})}$**.** As C_d is dimensionless

$$C_d = f\left[\frac{D}{H}, \frac{k}{H}, (\sigma/\rho\, gH^2), \frac{\mu}{(\rho H\sqrt{gH})}\right]$$

Check the dimensions of these π terms.

Problem 11. *The volume flow rate of a gas through a sharp edged orifice is found to be influenced by the pressure drop, orifice diameter and density and kinematic viscosity of the gas. Using the method of dimensional analysis obtain an expression for the flow rate.*

The variables and dimensions are listed below, adopting MLT system

S.No.	Variable	Unit	Dimensions
1	Volume flow rate, Q	m^3/s	L^3/T
2	Pressure drop, ΔP	N/m^2	M/LT^2
3	Diameter, D	m	L
4	Density, ρ	kg/m^3	M/L^3
5	Kinematic viscosity, v	m^2/s	L^2/T

There are five parameters and three dimensions. So two π terms can be obtained. Choosing ΔP, D and ρ as repeating variables,

Let $\pi_1 = Q\,\Delta P^a D^b \rho^c$ or $M^0L^0T^0 = \dfrac{L^3}{T}\,\dfrac{M^a}{L^aT^{2a}}\,L^b\,\dfrac{M^c}{L^{3c}}$

$\therefore$ $a + c = 0,\ 3 - a + b - 3c = 0,\ -1 - 2a = 0,$ $\therefore$ $a = -(1/2),\ c = 1/2,\ b = -2$

$\therefore$ $\boldsymbol{\pi_1 = (Q/D^2)\,(\rho/\Delta P)^{1/2}}$

Let $\qquad \pi_2 = v\Delta P^a D^b \rho^c \quad \text{or} \quad M^0L^0T^0 = \dfrac{L^2}{T}\,\dfrac{M^a}{L^aT^{2a}}\,L^b\,\dfrac{M^c}{L^{3c}}$

$\therefore \qquad a + c = 0,\ 2 - a + b - 3c = 0,\ -1 - 2a = 0,$

$\therefore \qquad a = -(1/2),\ c = (1/2),\ b = -1$

$\therefore \qquad \boldsymbol{\pi_2 = (v/D)\ (\rho/\Delta P)^{1/2}} \quad \text{or} \quad \dfrac{Q}{D^2}\left(\dfrac{\rho}{\Delta P}\right)^{1/2} = f\left[\dfrac{v}{D}\left(\dfrac{\rho}{\Delta P}\right)^{1/2}\right]$

Note : π_2 can be also identified as Reynolds number. Try to verify.

Problem.12 *In flow through a sudden contraction in a circular duct the head loss h is found to depend on the inlet velocity u, diameters D and d and the fluid properties density* ρ *viscosity* μ *and gravitational acceleration, g. Determine dimensionless parameters to correlate experimental results.*

The influencing variables with dimensions are tabulated below with *MLT* set.

S.No.	Variable	Unit	Dimensions
1	Loss of head, h	m	L
2	Inlet diameter, D	m	L
3	Outlet diameter, d	m	L
4	Velocity, u	m/s	L/T
5	Density, ρ	kg/m^3	M/L^3
6	Viscosity, μ	kg/ms	M/LT
7	Gravitational acceleration, g	m/s^2	L/T^2

There are seven variables and three dimensions. Hence four π parameters can be found. Considering D, ρ and u as repeating variables,

Let $\qquad \pi_1 = h\,D^a\,\rho^b\,u^c \quad \text{or} \quad M^0L^0T^0 = LL^a\,\dfrac{M^b}{L^{3b}}\,\dfrac{L^c}{T^c}$

$\therefore \qquad b = 0,\ 1 + a - 3b + c = 0,\ c = 0 \quad \therefore \quad a = -1 \quad \therefore \quad \boldsymbol{\pi_1 = h/D}$

Let $\qquad \pi_2 = d\,D^a\,\rho^b\,u^c \quad \text{or} \quad M^0L^0T^0 = LL^a\,\dfrac{M^b}{L^{3b}}\,\dfrac{L^c}{T^c}$

$\therefore \qquad b = 0,\ c = 0,\ 1 + a - 3b + c = 0,\ a = -1 \quad \therefore \quad \pi_2 = \mathbf{d/D}$

Let $\qquad \pi_3 = \mu\,D^a\,\rho^b\,u^c \quad \text{or} \quad M^0L^0T^0 = \dfrac{M}{LT}\,L^a\,\dfrac{M^b}{L^{3b}}\,\dfrac{L^c}{T^c}$

$\therefore \qquad b + 1 = 0,\ -1 + a - 3b + c = 0,\ -1 - c = 0,$

$\therefore \qquad b = -1,\ c = -1,\ a = -1$

$\therefore \qquad \boldsymbol{\pi_3 = \mu/D\rho u \quad \text{or} \quad \rho Du/\mu}$

Let $\qquad \pi_4 = g\,D^a\,\rho^b\,u^c \quad \text{or} \quad M^0L^0T^0 = \dfrac{L}{T^2}\,L^a\,\dfrac{M^b}{L^{3b}}\,\dfrac{L^c}{T^c}$

$\therefore \qquad b = 0, 1 + a - 3b + c = 0, -2 - c = 0$

$\therefore \qquad c = -2, a = 1 \quad \therefore \quad \boldsymbol{\pi_4 = gD/u^2}$

$$\therefore \qquad \frac{h}{D} = f\left[\frac{d}{D}, \frac{\rho D u}{\mu}, \frac{gD}{u^2}\right]$$

Note : gD/u^2 is the ratio of Potential energy to Kinetic energy.

Problem 13. *The volume flow rate, Q over a V-notch depends on fluid properties namely density* ρ, *kinematic viscosity v, and surface tension* σ. *It is also influenced by the angle of the notch, head of fluid over the vertex, and acceleration due to gravity. Determine the dimensionless parameters which can correlate the variables.*

As θ, the notch angle is a dimensionless parameter, the other parameters are listed below with dimensions, adopting *MLT* set.

S.No.	Variable	Unit	Dimension
1	Density, ρ	kg/m^3	M/L^3
2	kinematic vicosity, v	m^2/s	L^2/T
3	Surface tension, σ	N/m	M/T^2
4	Head of fluid, h	m	L
5	Gravitational acceleration, g	m/s^2	L/T^2
6	Flow rate, Q	m^3/s	L^3/T

There are six parameters and three dimensions. So three π terms can be identified. Considering ρ, g and h as repeating variables.

Let $\qquad \pi_1 = Q\, \rho^a\, g^b\, h^c \quad \text{or} \quad M^0L^0T^0 = \frac{L^3}{T}\frac{M^a}{L^{3a}}\frac{L^b}{T^{2b}}\, L^c$

$\therefore \qquad a = 0, 3 - 3a + b + c = 0, -1 - 2b = 0$

$\therefore \qquad b = -0.5, c = 2.5 \quad \therefore \quad \boldsymbol{\pi_1 = Q/g^{1/2}\, h^{5/2}}$

Let $\qquad \pi_2 = v\rho^a\, g^b\, h^c \quad \text{or} \quad M^0L^0T^0 = \frac{L^2}{T}\frac{M^a}{L^{3a}}\frac{L^b}{T^{2b}}\, L^c$

$\therefore \qquad a = 0, 2 - 3a + b + c = 0, -1 - 2b = 0$

$\therefore \qquad b = -0.5, c = (-1.5) \quad \therefore \quad \boldsymbol{\pi_2 = v/g^{1/2}\, h^{3/2}}$

Let $\qquad \pi_3 = \sigma\, \rho^a\, g^b\, h^c \quad \text{or} \quad M^0L^0T^0 = \frac{M}{T^2}\frac{M^a}{L^{3a}}\frac{L^b}{T^{2b}}\, L^c$

$\therefore \qquad 1 + a = 0, -3a + b + c = 0, -2 - 2b = 0 \quad \therefore \quad a = -1, b = -1, c = -2$

$$\therefore \qquad \boldsymbol{\pi_3 = \sigma/\rho g h^2} \quad \therefore \quad Q = g^{1/2}\, h^{5/2} f\left[\frac{v}{g^{1/2}\, h^{3/2}}, \frac{\sigma}{\rho\, g h^2}, \theta\right]$$

Note : In case surface tension is not considered, π_3 will not exist. π_2 can be identified as Reynolds number.

Problem 14. *The capillary rise h is found to be influenced by the tube diameter D, density* ρ, *gravitational acceleration g and surface tension* σ. *Determine the dimensionless parameters for the correlation of experimental results.*

The variables are listed below adopting *MLT* set of dimensions.

S.No.	Variable	Unit	Dimension
1	Diameter, D	m	L
2	Density, ρ	kg/m^3	M/L^3
3	Gravitational acceleration, g	m/s^2	L/T^2
4	Surface tension, σ	N/m	M/T^2
5	Capillary rise, h	m	L

There are five parameters and three dimensions and so two π parameters can be identified. Considering D, ρ and g as repeating variables,

Let $\quad \pi_1 = h\, D^a\, \rho^b\, g^c \quad \text{or} \quad M^0L^0T^0 = LL^a \dfrac{M^b}{L^{3b}} \dfrac{L^c}{T^{2c}}$

$\therefore \quad b = 0,\ 1 + a - 3b + c = 0,\ -2c = 0$

$\therefore \quad a = -1,\ b = 0,\ c = 0 \quad \therefore \quad \boldsymbol{\pi_1 = h/D}$

Let $\quad \pi_2 = \sigma\, D^a \rho^b\, g^c \quad \text{or} \quad M^0L^0T^0 = \dfrac{M}{T^2} L^a \dfrac{M^b}{L^{3b}} \dfrac{L^c}{T^{2c}}$

$\therefore \quad 1 + b = 0,\ a - 3b + c = 0,\ -2 - 2c = 0 \quad \therefore \quad b = -1,\ c = -1, \text{ and } a = -2$

$\therefore \quad \boldsymbol{\pi_2 = \sigma/D^2 g\rho}$, $g\rho$ can also be considered as specific weight γ

$$\frac{h}{D} = f\left[\frac{\sigma}{D^2\gamma}\right],$$

Note : π_2 can be identified as 1/Weber number.

Problems 15. *Show that the power P, developed by a hydraulic turbine can be correlated by the dimensionless parameters* $P/\rho N^3D^5$ *and* N^2D^2/gh, *where* ρ *is the density of water and N is the rotational speed, D is the runner diameter, h is the head and g is acceleration due to gravity.*

The parameters with dimensions are tabulated below using *MLT* set.

S.No.	Variable	Unit	Dimension
1	Power, P	W	ML^2/T^3
2	Density, ρ	kg/m^3	M/L^3
3	Speed, N	1/s	$1/T$
4	Diameter, D	m	L
5	Head, h	m	L
6	Gravitational acceleration, g	m/s^2	L/T^2

There are six parameters and three dimensions. So three π terms can be found. Choosing ρ, D and N as repeating variables,

Let $\pi_1 = P\,\rho^a D^b N^c$ or $M^0L^0T^0 = \dfrac{ML^2}{T^3}\dfrac{M^a}{L^{3a}} L^b \dfrac{1}{T^c}$

$\therefore$ $1 + a = 0,\ 2 - 3a + b = 0,\ -3 - c = 0 \quad \therefore \quad a = -1,\ c = -3,\ b = -5$

$\therefore$ $\boldsymbol{\pi_1 = P/\rho N^3 D^5}$ **(Power coefficient)**

Let $\pi_2 = h\,\rho^a D^b N^c$ or $M^0L^0T^0 = L\dfrac{M^a}{L^{3a}} L^b \dfrac{1}{T^c}$

$\therefore$ $a = 0,\ 1 - 3a + b = 0,\ c = 0, \quad \therefore \quad b = -1 \quad \therefore \quad \boldsymbol{\pi_2 = h/D.}$

Let $\pi_3 = g\,\rho^a D^b N^c$ or $M^0L^0T^0 = \dfrac{L}{T^2}\dfrac{M^a}{L^{3a}} L^b \dfrac{1}{T^c}$

$\therefore$ $a = 0,\ 1 - 3a + b = 0,\ -2 - c = 0$

$\therefore$ $c = -2,\ b = -1 \quad \therefore \quad \boldsymbol{\pi_3 = g/DN^2}$

$\boldsymbol{\pi_2 \times \pi_3 = gh/D^2N^2}$ **(Head coefficient)**

$$\therefore \qquad \frac{P}{\rho N^3 D^5} = f\left[\frac{gh}{D^2N^2}\right]$$

In this expression the first term is called power coefficient and the second one is called head coefficient. These are used in model testing of turbo machines.

Problem 16. *The power developed by hydraulic machines is found to depend on the head h, flow rate Q, density* ρ*, speed N, runner diameter D, and acceleration due to gravity, g. Obtain suitable dimensionless parameters to correlate experimental results.*

The parameters with dimensions are listed below, adopting *MLT* set of dimensions.

S.No.	Variable	Unit	Dimension
1	Power, P	W	ML^2/T^3
2	Head, h	m	L
3	Flow rate, Q	m^3/s	L^3/T
4	Density, ρ	kg/m^3	M/L^3
5	Speed, N	1/s	$1/T$
6	Diameter, D	m	L
7	Acceleration due to gravity, g	m/s^2	L/T^2

There are seven variables and three dimensions and hence four π terms can be formed. Taking ρ, D and g as repeating variables.

Let $\pi_1 = P\,\rho^a D^b g^c$ or $M^0L^0T^0 = \dfrac{ML^2}{T^3}\dfrac{M^a}{L^{3a}} L^b \dfrac{L^c}{T^{2c}}$

$\therefore \quad 1 + a = 0,\ 2 - 3a + b + c = 0,\ -3 - 2c = 0$

$\therefore \quad a = -1,\ c = -3/2 \quad \therefore \quad b = -7/2$

$\therefore \quad \boldsymbol{\pi_1 = P/\rho D^{7/2}\, g^{3/2}}$

Let $\quad \pi_2 = h\, \rho^a\, D^b\, g^c \quad$ or $\quad M^0L^0T^0 = L\, \dfrac{M^a}{L^{3a}}\, L^b\, \dfrac{L^c}{T^{2c}}$

$\therefore \quad a = 0,\ 1 - 3a + b + c = 0,\ -2c = 0$

$\therefore \quad a = 0,\ b = -1,\ c = 0 \quad \therefore \quad \boldsymbol{\pi_2 = h/D}$

Let $\quad \pi_3 = Q\, \rho^a\, D^b\, g^c \quad$ or $\quad M^0L^0T^0 = \dfrac{L^3}{T}\, \dfrac{M^a}{L^{3a}}\, L^b\, \dfrac{L^c}{T^{2c}}$

$\therefore \quad a = 0,\ 3 - 3a + b + c = 0,\ -1 - 2c = 0$

$\therefore \quad a = 0,\ c = -1/2,\ b = -5/2$

$\therefore \quad \boldsymbol{\pi_3 = Q/g^{1/2}\, D^{5/2}}$

Let $\quad \pi_4 = N\, \rho^a\, D^b\, g^c \quad$ or $\quad M^0L^0T^0 = \dfrac{1}{T}\, \dfrac{M^a}{L^{3a}}\, L^b\, \dfrac{L^c}{T^{2c}}$

$\therefore \quad a = 0,\ -3a + b + c = 0,\ -1 - 2c = 0 \quad \therefore \quad a = 0,\ c = -1/2,\ b = 1/2$

$\therefore \quad \boldsymbol{\pi_4 = ND^{1/2}/g^{1/2}}$

The coefficients popularly used in model testing are given below. These can be obtained from the above four π terms.

1. Head coefficient $\dfrac{gh}{N^2D^2} = \dfrac{\pi_2}{\pi_4{}^2} = \dfrac{hg}{DN^2\, D} = \dfrac{gh}{N^2D^2}$

2. Flow coefficient $\dfrac{Q}{ND^3} = \dfrac{\pi_3}{\pi_4} = \dfrac{Qg^{1/2}}{g^{1/2}\, D^{5/2}\, ND^{1/2}} = \dfrac{Q}{N\, D^3}$

3. Power coefficient $\dfrac{P}{\rho N^3\, D^5} = \dfrac{\pi_1}{\pi_4{}^3} = \dfrac{Pg^{3/2}}{\rho D^{7/2}\, g^{3/2}\, N^3\, D^{3/2}} = \dfrac{P}{\rho\, N^3 D^5}$

4. Specific speed based on Q, for pumps, $\mathbf{N_{sp}}$

$$\frac{N\sqrt{Q}}{(gh)^{3/4}} = \frac{(\text{flow coeff})^{1/2}}{(\text{head coeff})^{3/4}} = \frac{Q^{1/2}}{N^{1/2}D^{3/2}}\, \frac{N^{3/2}D^{3/2}}{(gh)^{3/4}} = \frac{N\sqrt{Q}}{(gh)^{3/4}}$$

(dimensional specific speed $N\sqrt{Q}/h^{3/4}$ is commonly used as mostly water is the fluid used)

5. Specific speed based on power, for Turbines

$$N_{st} = \frac{N\sqrt{P}}{\rho^{1/2}\, (gh)^{5/4}} = \frac{(\text{power coefficient})^{1/2}}{(\text{head coefficient})^{5/4}} = \frac{P^{1/2}(ND)^{5/2}}{\rho^{1/2}\, N^{3/2}\, D^{5/2}\, (gh)^{5/4}} = \frac{N\sqrt{P}}{\rho^{1/2}\, (gh)^{5/4}}$$

$$\left(\text{Dimensional Specific speed } \frac{N\sqrt{P}}{h^{5/4}} \text{ is commonly used as water is used in most cases}\right)$$

These are the popularly used dimensionally numbers in hydraulic turbo machinery.

Problem 17. *In forced convection in pipes heat transfer coefficient h is found to depend on thermal conductivity, viscosity, density, specific heat, flow velocity and the diameter. Obtain dimensionless parameters to correlate experimental results.*

The variables with dimensions are listed below using $MLT\,\theta$ set of dimensions, where θ is temperature.

S.No.	Variable	Unit	Dimension
1	Convection coefficient, h	W/m^{2}K	$M/T^3\,\theta$
2	Diameter D	m	L
3	Thermal conductivity, k	W/mK	$ML/T^3\,\theta$
4	Density, ρ	kg/m^3	M/L^3
5	Viscosity, μ	kg/ms	M/LT
6	Specific heat, c	Nm/kgK	$L^2/T^2\,\theta$
7	Flow velocity, u	m/s	L/T

Three π terms are possible as there are seven variables and four dimensions. Choosing k, μ, ρ and D as repeating variables.

Let $\pi_1 = h\,k^a\,\mu^b\,\rho^c\,D^d$ or $M^0L^0T^0\,\theta^0 = \dfrac{M}{T^3\theta}\,\dfrac{M^a}{T^{3a}}\,\dfrac{L^a}{\theta^a}\,\dfrac{M^b}{L^bT^b}\,\dfrac{M^c}{L^{3c}}\,L^d$

$\therefore$ $1 + a + b + c = 0,\ a - b - 3c + d = 0,\ -3 - 3a - b = 0,\ -1 - a = 0$

$\therefore$ $a = -1,\ b = 0,\ c = 0,\ d = 1$

$\therefore$ $\boldsymbol{\pi_1 = hD/k}$ **(Nusselt number)**

Let $\pi_2 = u\,k^a\,\mu^b\,\rho^c\,D^d$ or $M^0L^0T^0\,\theta^0 = \dfrac{L}{T}\,\dfrac{M^a}{T^{3a}}\,\dfrac{L^a}{\theta^a}\,\dfrac{M^b}{L^bT^b}\,\dfrac{M^c}{L^{3c}}\,L^d$

$\therefore$ $a + b + c = 0,\ 1 + a - b - 3c + d = 0,\ -1 - 3a - b = 0,\ -a = 0,$

$\therefore$ $a = 0,\ b = -1,\ c = 1,\ d = 1$

$\therefore$ $\boldsymbol{\pi_2 = u\rho D/\mu}$ **(Reynolds number)**

Let $\pi_3 = ck^a\,\mu^b\,\rho^c\,D^d$ or $M^0L^0T^0\,\theta^0 = \dfrac{L^2}{T^2\theta}\,\dfrac{M^a}{T^{3a}}\,\dfrac{L^a}{\theta^a}\,\dfrac{M^b}{L^bT^b}\,\dfrac{M^c}{L^{3c}}\,L^d$

$\therefore$ $a + b + c = 0,\ 2 + a - b - 3c + d = 0,\ -2 - 3a - b = 0,\ -1 - a = 0,$

$\therefore$ $a = -1,\ b = 1,\ c = 0,\ d = 0$

$\therefore$ $\boldsymbol{\pi_3 = c\mu/k}$ **(Prandtl number),** $\dfrac{hD}{k} = f\left[\dfrac{uD\rho}{\mu}, \dfrac{c\mu}{k}\right]$

These are popular dimensionless numbers in convective heat transfer.

Problem 18. *The temperature difference θ at a location x at time τ in a slab of thickness L originally at a temperature difference θ_0 with outside is found to depend on the thermal diffusivity α, thermal conductivity k and convection coefficient h. Using dimensional analysis determine the dimensionless parameters to correlate the situation.*

The influencing parameters with dimensions are listed below, choosing $MLT\,\theta$ set.

S.No.	Parameter	Unit	Dimension
1	Slab thickness, L	m	L
2	Location distance, x	m	L
3	Initial temperature difference, θ_0	deg K	θ_0
4	Temperature difference at time τ, θ	deg K	θ
5	Time, τ	s	T
6	Thermal diffusivity, α	m^2/s	L^2/T
7	Thermal conductivity, k	W/mK	$ML/T^3\theta$
8	Convection coefficient, h	W/m^2 K	$M/T^3\theta$

There are eight variables and four dimensions. Hence four π terms can be identified. Choosing θ_0, L, α and k as repeating variables,

Let $\pi_1 = \theta\, \theta_0^a\, L^b\, \alpha^c\, k^d$ or $M^0L^0T^0\theta^0 = \theta\, \theta^a L^b \dfrac{L^{2c}}{T^c} \dfrac{M^d\, L^d}{T^{3d}\, \theta^d}$

$\therefore$ $d = 0,\ b + 2c + d = 0,\ -c - 3d = 0,\ \ 1 + a - d = 0,$

$\therefore$ $d = 0,\ c = 0,\ b = 0,\ a = -1$

$\therefore$ $\boldsymbol{\pi_1 = \theta/\theta_0}$

Let $\pi_2 = x\, \theta_0^a\, L^b\, \alpha^c\, k^d$ or $M^0L^0T^0\theta^0 = L\theta^a L^b \dfrac{L^{2c}}{T^c} \dfrac{M^d\, L^d}{T^{3d}\, \theta^d}$

$\therefore$ $a - d = 0,\ d = 0,\ 1 + b + 2c + d = 0,\ -c - 3d = 0,$

$\therefore$ $a = 0,\ b = -1,\ c = 0,\ d = 0$

$\therefore$ $\boldsymbol{\pi_2 = x/L}$

Let $\pi_3 = h\, \theta_0^a\, L^b\, \alpha^c\, k^d$ or $M^0L^0T^0\theta^0 = \dfrac{M}{T^3\theta}\, \theta^a L^b \dfrac{L^{2c}}{T^c} \dfrac{M^d\, L^d}{T^{3d}\, \theta^d}$

$\therefore$ $1 + d = 0,\ b + 2c + d = 0,\ \ -3 - c - 3d = 0,\ -1 + a - d = 0,$

$\therefore$ $a = 0,\ b = 1,\ c = 0,\ d = -1$

$\therefore$ $\boldsymbol{\pi_3 = hL/k}$ **(Biot number)**

Let $\pi_4 = \tau\, \theta_0^a\, L^b\, \alpha^c\, k^d$ or $M^0L^0T^0\theta^0 = T\, \theta^a\, L^b \dfrac{L^{2c}}{T^c} \dfrac{M^d\, L^d}{T^{3d}\, \theta^d}$

$\therefore$ $d = 0,\ b + 2c + d = 0,\ 1 - c - 3d = 0,\ a - d = 0,$

$\therefore$ $a = 0,\ b = -2,\ c = 1,\ d = 0$

$\therefore$ $\boldsymbol{\pi_4 = \alpha\, \tau/L^2}$ **(Fourier number)**

$$\therefore \qquad \frac{\theta}{\theta_0} = f\left[\frac{x}{L}, \frac{hL}{k}, \frac{\alpha\tau}{L^2}\right]$$

There are the popular dimensionless numbers is conduction heat transfer.

This problem shows that the method is not limited to fluid flow or convection only.

Problem 19. *Convective heat transfer coefficient in free convection over a surface is found to be influenced by the density, viscosity, thermal conductivity, coefficient of cubical expansion, temperature difference, gravitational acceleration, specific heat, the height of surface and the flow velocity. Using dimensional analysis, determine the dimensionless parameters that will correlate the phenomenon.*

The variables with dimensions in the $MLT\,\theta$ set is tabulated below.

S.No.	Variable	Unit	Dimension
1	Height, x	m	L
2	Temperature difference, ΔT	deg K	θ
3	Coefficient of cubical expansion, β	(m^3/m^3)/deg K	$1/\theta$
4	Acceleration due to gravity, g	m/s^2	L/T^2
5	Density, ρ	kg/m^3	M/L^3
6	Viscosity, μ	kg/ms	M/LT
7	Specific heat, c	J/kgK	$L^2/T^2\theta$
8	Thermal conductivity, k	W/mK	$ML/T^3\theta$
9	Convective heat transfer coefficient, h	W/m^2K	$M/T^3\theta$

There are nine variables and four dimensions. Hence five π terms can be identified. ρ, μ, x and k are chosen as repeating variables.

Let $\qquad \pi_1 = \Delta T\, \rho^a\, \mu^b x^c k^d \quad \text{or} \quad M^0L^0T^0\theta^0 = \theta\, \frac{M^a}{L^{3a}}\, \frac{M^b}{L^bT^b}\, L^c\, \frac{M^dL^d}{T^{3d}\theta^d}$

$\therefore \qquad a + b + d = 0, -3a - b + c + d = 0, b - 3d = 0, 1 - d = 0,$

$\therefore \qquad a = 2, b = -3, c = 2, d = 1$

$\therefore \qquad \boldsymbol{\pi_1 = \Delta T\rho^2\, x^2\, k/\mu^3}$

Let $\qquad \pi_2 = \beta\, \rho^a\, \mu^b x^c k^d \quad \text{or} \quad M^0L^0T^0\theta^0 = \frac{1}{\theta}\, \frac{M^a}{L^{3a}}\, \frac{M^b}{L^bT^b}\, L^c\, \frac{M^dL^d}{T^{3d}\theta^d}$

$\therefore \qquad a + b + d = 0, -3a - b + c + d = 0, \quad -b - 3d = 0, -1 - d = 0,$

$\therefore \qquad a = -2, b = 3, c = -2, d = -1$

$\therefore \qquad \boldsymbol{\pi_2 = \beta\, \mu^3/\rho^2x^2k}$

Let $\qquad \pi_3 = g\, \rho^a\, \mu^b x^c k^d \quad \text{or} \quad M^0L^0T^0\theta^0 = \frac{L}{T^2}\, \frac{M^a}{L^{3a}}\, \frac{M^b}{L^bT^b}\, L^c\, \frac{M^dL^d}{T^{3d}\theta^d}$

$\therefore \quad a + b + d = 0,\ 1 - 3a - b + c + d = 0,\ -2 - b - 3d = 0,\ d = 0,$

$\therefore \quad a = 2,\ b = -2,\ c = 3,\ d = 0$

$\therefore \quad \boldsymbol{\pi_3 = g\,\rho^2\,x^3/\mu^2}$

Let $\quad \pi_4 = c\,\rho^a\,\mu^b x^c k^d \quad \text{or} \quad M^0L^0T^0\theta^0 = \dfrac{L^2}{T^2\theta}\,\dfrac{M^a}{L^{3a}}\,\dfrac{M^b}{L^bT^b}\,L^c\,\dfrac{M^dL^d}{T^{3d}\theta^d}$

$\therefore \quad a + b + d = 0,\ 2 - 3a - b + c + d = 0,\ -2 - b - 3d = 0,\ -1 - d = 0,$

$\therefore \quad a = 0,\ b = 1,\ c = 0,\ d = -1$

$\therefore \quad \boldsymbol{\pi_4 = c\,\mu/k}$ **(Prandtl number)**

Let $\quad \pi_5 = h\,\rho^a\,\mu^b x^c k^d \quad \text{or} \quad M^0L^0T^0\theta^0 = \dfrac{M}{T^3\theta}\,\dfrac{M^a}{L^{3a}}\,\dfrac{M^b}{L^bT^b}\,L^c\,\dfrac{M^dL^d}{T^{3d}\theta^d}$

$\therefore \quad 1 + a + b + d = 0,\ -3a - b + c + d = 0,\ -3 - b - 3d = 0,\ -1 - d = 0,$

$\therefore \quad a = 0,\ b = 0,\ c = 1,\ d = -1$

$\therefore \quad \boldsymbol{\pi_5 = hx/k}$ **(Nusselt number)**

As the π terms are too many π_1, π_2 and π_3 are combined as $\pi_1 \times \pi_2 \times \pi_3$ to form the group known as Grashof number.

$$\boldsymbol{\pi_6} = \frac{\Delta T\rho^2x^2k}{\mu^3} \times \frac{\beta\mu^3}{\rho^2x^2k} \times \frac{g\rho^2x^3}{\mu^2} = \frac{\Delta T\beta g x^3\rho^2}{\mu^2} = \frac{\Delta T g\beta x^3}{v^2}$$

$$\therefore \quad \frac{hx}{k} = f\left[\frac{c\mu}{k}, \frac{\Delta T g\beta\rho^2x^3}{\mu^2}\right]$$

Note : When there are more than three π parameters the set should be reduced to three by judicial combination.

OBJECTIVE QUESTIONS

O Q. 8.1. Fill in the blanks:

1. The dimension for force in the *MLT* set is ________________.
2. The dimension for mass in the *FLT* set is ________________.
3. If there are *n* variables and *m* dimensions, π-theorem states that ________________ dimensionless parameters can be obtained.
4. The dimension for thermal conductivity in the *MLT* θ system is ________________ and in *FLT* θ system is ________________.
5. For an expression to be dimensionally homogeneous, each additive term in the equations should have ________________.
6. One of the methods to check the correctness of an equation is to check for ________________ for each of the additive terms.
7. The limitation of dimensional analysis is that the ________________ has to be determined by experiments.
8. The approximate number of experiments to evaluate the influence of 5 parameters separately is ________________ assuming that 10 experiments are needed for each variable.

Chapter-5 Similitude and Model Testing

5.0 INTRODUCTION

Fluid flow analysis is involved in the design of aircrafts, ships, submarines, turbines, pumps, harbours and tall buildings and structures. Fluid flow is influenced by several factors and because of this the analysis is more complex. For many practical situations exact soluations are not available. The estimates may vary by as much as ± 20%. Because of this it is not possible to rely solely on design calculations and performance predictions. **Experimental validation of the design is thus found necessary.** Consider the case of a hydraulic turbine of 50 MW size. It will be a very costly failure if the design performance and the actual performance differ. If we can predict its performance before manufacturing the unit it will be very useful. Model testing comes to our aid in this situation. **Constructing and testing small versions of the unit is called model testing. Similarity of features enable the prediction of the performance** of the full size unit from the test results of the smaller unit. The application of dimensional analysis is helpful in planning of the experiments as well as prediction of the performance of the larger unit from the test results of the model.

5.1 MODEL AND PROTOTYPE

In the engineering point of view model can be defined as the representation of physical system that may be used to predict the behavior of the system in the desired aspect. **The system whose behavior is to be predicted by the model is called the prototype**. The discussion in this chapter is about physical models that resemble the prototype but are generally smaller in size. These may also operate with different fluids, at different pressures, velocities etc. As models are generally smaller than the prototype, these are cheaper to build and test. Model testing is also used for evaluating proposed modifications to existing systems. The effect of the changes on the performance of the system can be predicted by model testing before attempting the modifications. Models should be carefully designed for reliable prediction of the prototype performance.

5.2 CONDITIONS FOR SIMILARITY BETWEEN MODELS AND PROTOTYPE

Dimensional analysis provides a good basis for laying down the conditions for similarity. The *PI* theorem shows that the performance of any system (prototype) can be described by a functional relationship of the form given in equation 9.2.1.

$$\pi_{1p} = f(\pi_{2p}, \pi_{3p} \ldots\ldots \pi_{np}) \quad \ldots(5.2.1)$$

The *PI* terms include all the parameters influencing the system and are generally ratios of forces, lengths, energy etc. If a model is to be similar to the prototype and also function similarly as the prototype, then the *PI* terms for the model should also have the same value as that of the prototype or the same functional relationship as the prototype. (eqn. 9.2.1)

$$\pi_{1m} = f(\pi_{2m}, \pi_{3m} \ldots\ldots \pi_{nm}) \quad \ldots(5.2.2)$$

For such a condition to be satisfied, the model should be constructed and operated such that simultaneously

$$\pi_{1m} = \pi_{1p}, \pi_{2m} = \pi_{2p}, \ldots\ldots \pi_{nm} = \pi_{np}$$

Equation 9.2.3 provides the model design conditions. It is also called similarity requirements or modelling laws.

5.2.1 Geometric Similarity

Some of the *PI* terms involve the ratio of length parameters. All the similar linear dimension of the model and prototype should have the same ratio. This is called geometric similarity. The ratio is generally denoted by the scale or scale factor. One tenth scale model means that the similar linear dimensions of the model is 1/10 th of that of the prototype. For complete similarity all the linear dimensions of the model should bear the same ratio to those of the prototype. There are some situations where it is difficult to obtain such similarity. Roughness is one such case. In cases like ship, harbour or dams distorted models only are possible. In these cases the depth scale is different from length scale. Interpretation of the results of the tests on distorted models should be very carefully done. Geometric scale cannot be chosen without reference to other parameters. For example the choice of the scale when applied to the Reynolds number may dictate a very high velocity which may be difficult to achieve at a reasonable cost.

5.2.2 Dynamic Similarity

Similitude requires that π terms like Reynolds number, Froude number, Weber number etc. be equal for the model and prototype. These numbers are ratios of inertia, viscous gravity and surface tension forces. This condition implies that the ratio of forces on fluid elements at corresponding points (homologous) in the model and prototype should be the same. This requirement is called dynamic similarity. This is a basic requirement in model design. If model and prototype are dynamically similar then the performance of the prototype can be predicted from the measurements on the model. In some cases it may be difficult to hold simultaneously equality of two dimensionless numbers. In such situations, the parameter having a larger influence on the performance may have to be chosen. This happens for example in the case of model tasting of ships. Both Reynolds number and Froude number should be simultaneously held equal between the model and prototype. This is not possible as this would require either fluids with a very large difference in their viscosities or the use of very large velocities with the model. This is illustrated in problem 9.14.

5.2.3 Kinematic Similarity

When both geometric and dynamic similarities exist, then velocity ratios and acceleration ratios will be the same throughout the flow field. This will mean that the streamline patterns will be the same in both cases of model and prototype. This is called kinematic similarly. **To achieve complete similarity between model and prototype all the three similarities - geometric, dynamic and kinematic should be maintained.**

5.3 TYPES OF MODEL STUDIES

Model testing can be broadly classified on the basis of the general nature of flow into four types. These are

(1) Flow through closed conduits

(2) Flow around immersed bodies

(3) Flow with free surface and

(4) Flow through turbomachinery

5.3.1 Flow through Closed Conduits

Flow through pipes, valves, fittings and measuring devices are dealt under this category. The conduits are generally circular, but there may be changes along the flow direction. As the wall shear is an important force, Reynolds number is the most important parameter. The pressure drop along the flow is more often the required parameter to be evaluated. Compressibility effect is negligible at low mach numbers. (M<0.3).

From dimensional analysis the pressure drop can be established as

$$\Delta P/\rho u^2 = f\left(\frac{\rho u L}{\mu}, \frac{\varepsilon}{L}, \frac{D}{L}\right) \qquad ...(5.3.1)$$

The geometric scale is given by the ratio, scale = L_m/L_p.

This requires $\dfrac{D_m}{D_p} = \dfrac{\varepsilon_m}{\varepsilon_p} = \dfrac{L_m}{L_p} = \lambda.$

Reynolds number similarity leads to the condition for velocity ratio as

$$\frac{u_m \rho_m L_m}{\mu_m} = \frac{u_p \rho_p L_p}{\mu_p} \quad \therefore \quad \frac{u_m}{u_p} = \frac{\mu_m}{\mu_p}\,\frac{\rho_p}{\rho_m}\,\frac{L_p}{L_m}$$

If the fluid used for the model and prototype are the same, then $\dfrac{u_m}{u_p} = \dfrac{L_p}{L_m}$ or $u_m = u_p/\lambda$.

As λ is less than one, the velocity to be used with the model has to be higher compared to the prototype. Otherwise a different fluid with higher viscosity should be chosen to satisfy the requirements.

The pressure drop in the prototype is calculated as in equation (9.3.3)

From equality of, $\Delta P / \rho u^2$, $\Delta P_P = \dfrac{\rho_p}{\rho_m}\left(\dfrac{u_p}{u_m}\right)^2 \Delta P_m$

As ΔP_m is measured, using the model, the pressure drop in the prototype can be predicted.

When Reynolds numbers are large the inertia forces are predominant and viscous forces will be small in comparison. In such cases, the Reynolds number similarity becomes unimportant. However, the model should be tested at various Reynolds numbers to determine the range at which its effect on pressure drop becomes negligible. After this is established the model test results can be applied without regard to Reynolds number similarity, in this range.

Another condition is the onset of cavitation at some locations in the flow, particularly in testing components where at some points the local velocity may become high and pressure may drop to a level where cavitation may set in. Unless cavitation effects are the aim of the study, such condition should be avoided. In case cavitation effects are to be studied, then similarity of cavitation number should be established. *i.e.* $(p_r - p_v)/(\rho u^2/2)$. Where p_r is the reference pressure and p_v is the vapour pressure at that temperature.

5.3.2 Flow Around Immersed Bodies

Aircraft, Submarine, cars and trucks and recently buildings are examples for this type of study. In the sports area golf and tennis balls are examples for this type of study. Models are usually tested in wind tunnels. As viscous forces over the surface and inertia forces on fluid elements are involved in this case also, Reynolds number of the model and prototype should be equal. Gravity and surface tension forces are not involved in this case and hence Froude and Weber numbers need not be considered. Drag coefficient, defined by [Drag force /(1/2) ρu^2 l^2)] is the desired quantity to be predicted. Generally the following relationship holds in this case.

$$C_D = \frac{D}{(1/2)\rho u^2 l^2} = f\left[\frac{l_1}{l}, \frac{\varepsilon}{l}, \frac{\rho u l}{\mu}\right] \tag{5.3.4}$$

where l is a characteristic length of the system and l_1 represents the other length parameter affecting the flow and ε is the roughness of the surface.

When the flow speed increases beyond Mach number 0.3 compressibility effect on similarity should be considered. Using the similitude, measured values of drag on model is used to estimate the drag on the prototype.

$$D_p = D_m \frac{\rho_p}{\rho_m}\left(\frac{u_p}{u_m} \times \frac{l_p}{l_m}\right)^2$$

From Reynolds number similitude

$$u_m = \frac{\mu_m}{\mu_p}\frac{\rho_p}{\rho_m}\frac{l_p}{l_m}u_p = \frac{v_m}{v_p}\frac{l_p}{l_m}u_p \tag{5.3.6}$$

When same fluid is used for both prototype and model

$$u_m = (l_p/l_m)\, u_p \qquad \text{...(5.3.7)}$$

The model velocity should be higher by the geometric scale.

If the prototype is to operate at 100 kmph and if the scale is 1:10, then the model should operate at 1000 kmph, which will mean a high Mach number. The model will be influenced by compressibility effect due to the operation at high Mach numbers. The prototype however will be operating at low Mach numbers where compressibility effect is negligible. Hence the performance prediction will be in error.

This may be overcome by using different fluids say water in place of air. Using equation 9.3.6, as kinematic viscosity of air is about 10 times that of water, the velocity will now be at a reasonable level. Another method is to pressurise the air in the wind tunnel, thus increasing the density, and reducing the required velocity of the model.

Where expense is of no consideration due to the requirement of utmost reliability as in space applications and development of new aircraft, full scale models are also used.

In some cases at higher ranges, the Reynolds number is found to have little influence on drag. Strict Reynolds similarity need not be used in such situations. The variation of drag due to variation in Reynolds number for cylinder and sphere is shown as plotted in Fig. 9.3.1. It may be seen that above Re = 10^4 the curve is flat. If the operation of the prototype will be at such a range, then Reynolds number equality will not be insisted for model testing.

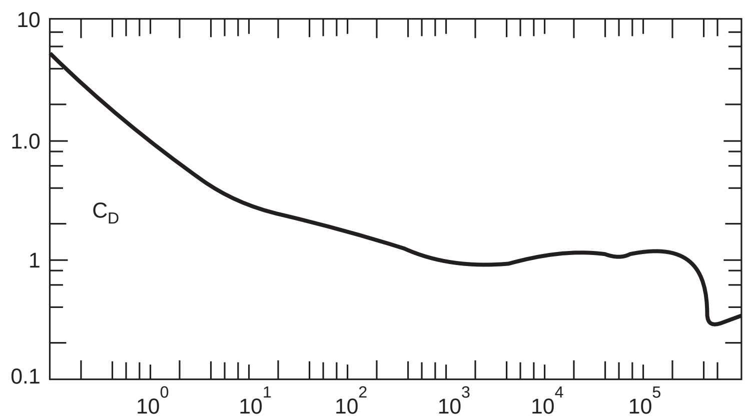


Figure 5.3.1 Variation of drag with Reynolds number for flow over cylinder

Another situation arises in testing of models of high speed aircraft. In this case the use of Mach number similitude requires equal velocities while the Reynolds number similarity requires increased velocity for the model as per geometric scale. In such cases distorted model is used to predict prototype performance.

5.3.3 Flow with Free Surface

Flow in canals, rivers as well as flow around ships come under this category. In these cases gravity and inertia forces are found to be governing the situation and hence Froude number becomes the main similarity parameter.

In some cases Weber number as well as Reynolds number may also influence the design of the model.

Considering Froude number, the velocity of the model is calculated as below.

$$\frac{u_m}{\sqrt{gl_m}} = \frac{u_p}{\sqrt{gl_p}}$$

$$\therefore \qquad u_m = u_p \sqrt{\frac{l_m}{l_p}} = u_p \sqrt{\text{scale}}$$

In case Reynolds number similarity has to be also considered, substituting this value of velocity ratio, the ratio of kinematic viscosities is given as

$$\frac{v_m}{v_p} = (\text{scale})^{3/2}$$

As these situations involve use of water in both model and prototype, it is impossible to satisfy the condition of equations 9.3.9 and 9.3.10 simultaneously. In such a case distorted model may have to be selected.

If surface tension also influences the flow, it is still more difficult to choose a fully similar model.

In many practical applications in this type of situation the influence of Weber and Reynolds number is rather small. Hence generally models are designed on the basis of Froude number similarity.

A special situation arises in the case of ships. The total drag on the ship as it moves is made up of two components: (1) The viscous shearing stress along the hull, (2) Pressure induced drag due to wave motion and influenced by the shape of the hull.

As it is not possible to build and operate a model satisfying simultaneously the Reynolds number similarity and Froude number similarity ingenious methods have to be adopted to calculate the total drag. The total drag on the model is first measured by experiment. The shear drag is analytically determined and the pressure drag on the model is calculated by subtracting this value. The drag on the prototype is determined using Froude number similarity. The calculated value of viscous drag is then added to obtain the total drag.

In case of design of river model, if the same vertical and horizontal scales are used, the depth will be low for the model and surface tension effects should be considered. But the use of distorted model, (vertical scaling smaller than horizontal scaling) overcomes this problem.

5.3.4 Models for Turbomachinery

Pumps as well as turbines are included in the general term turbomachines. Pumps are power absorbing machines which increase the head of the fluid passing though them. Turbines are power generating machines which reduce the head of the fluid passing through them.

The operating variables of the machines are the flow rate Q, the power P and the speed N. The fluid properties are the density and viscosity. The machine parameters are the diameter and a characteristic length and the roughness of the flow surface. Power, head and efficiency can be expressed as functions of π terms

$$\text{Power} = f_1\left(\frac{l}{D}, \frac{\varepsilon}{D}, \frac{Q}{N\,D^3}, \frac{\rho N D^2}{\mu}\right) \tag{5.3.11}$$

The term ε / D is not important due to the various sharp corners in the machine. The dimensionless term involving power is defined as power coefficient, defined as $C_p = P/\rho\, N^3\, D^3$. The head coefficient is defined as $C_h = gh/N^2\, D^2$. The term $Q/N\, D^3$ is called flow coefficient. If two similar machines are operated with the same flow coefficient, the power and head coefficients will also be equal for the machines. This will then lead to the same efficiency. Combining flow and head coefficients in the case of pumps will give the dimensionless specific speed of the pump.

$$N_{sp} = \frac{N\sqrt{Q}}{(gh)^{3/4}}$$

Popularly used dimensional specific speed for pumps is defined as

$$N_{sp} = \frac{N\sqrt{Q}}{h^{3/4}}$$

In the case of turbines, combining power and flow coefficients, the specific speed is obtained as

$$N_{st} = \frac{N\sqrt{P}}{\rho^{1/2}(gh)^{5/4}}$$

Popularly used dimensional speed for turbines is

$$N_{st} = \frac{N\sqrt{P}}{h^{5/4}}$$

In model testing at a particular speed, the flow rate at various delivery heads can be measured. This can be used to predict the performance of the pump at other speeds using the various coefficients defined. The procedure for turbines will also be similar. The model can be run at a constant speed when the head is varied, the power and flow rate can be measured. The performance of the prototype can be predicted from the results of the tests on the geometrically similar model.

5.4 NONDIMENSIONALISING GOVERNING DIFFERENTIAL EQUATIONS

When differential equations describing the phenomenon is not available, the method of dimensional analysis is used to obtain similarity conditions. When differential equations describing the system are available, similarity parameters can be deduced by non dimensionalising the equations.

Consider the continuity and x directional momentum equations for two dimensional flow,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

The various quantities can be made dimensionless by dividing by reference quantities, as given below

$$u^* = \frac{u}{U},\ v^* = \frac{v}{V},\ P^* = \frac{p}{P_0},\ x^* = \frac{x}{L},\ y^* = \frac{y}{L},\ t^* = \frac{t}{\tau}$$

Then $$\frac{\partial u}{\partial x} = \frac{U}{L}\frac{\partial u^*}{\partial x^*},\qquad \frac{\partial^2 u}{\partial x^2} = \frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}$$

Similar method is used in the case of other terms.

Substituting, the momentum equation reduces to the form

$$\left[\frac{L}{\tau U}\right]\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial t^*}+v^*\frac{\partial u^*}{\partial t^*}=-\left[\frac{P_0}{\rho U^2}\right]\frac{\partial P^*}{\partial x^*}+\left[\frac{\mu}{\rho UL}\right]\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

It may be noted that the non dimensionalised equation is similar to the general equation except for the terms in square brackets. These are the similarity parameters thus identified.

$$\frac{L}{\tau U},\ \frac{P}{\rho U^2},\ \frac{\mu}{\rho UL}$$

In case gravity force is added, gL/U^2 will be identified. These are forms of Strouhal, Euler, Reynolds and Froude numbers. As the equation describes the general unsteady flow all the numbers are involved. If other forms of forces like surface tension is added. Weber number can be identified. If equations for compressible flow is used, Mach number can be obtained by a similar method.

5.5 CONCLUSION

In all the problems in this chapter on model testing the π terms identified in chapter 8 are used. Reference may be made to the problems in chapter 8. The discussions in this chapter is limited to basics. In actual model making and testing as well as interpretation of results many other finer details have to be considered for obtaining accurate predictions about the performance of the prototype.

SOLVED PROBLEMS

Problem 5.1 *To study the pressure drop in flow of water through a pipe, a model of scale 1/10 is used.* ***Determine the ratio of pressure drops*** *between model and prototype if water is used in the model. In case air is used determine the ratio of pressure drops.*

Case (i) Water flow in both model and prototype.

Reynolds number similarity is to be maintained.

$$\frac{u_m d_m \rho_m}{\mu_m} \frac{u_p d_p \rho_p}{\mu_p} \qquad \therefore \qquad \frac{u_m}{u_p} = \frac{\mu_m}{\mu_p} \times \frac{d_p}{d_m} \times \frac{\rho_p}{\rho_m}$$

As viscosity and density values are the same,

$$\frac{u_m}{u_p} = \frac{d_p}{d_m} = 10,$$

The pressure drop is obtained using pressure coefficient

$$[\Delta P/(1/2)\, \rho u^2]_m = [\Delta P/(1/2)\, \rho u^2]_p$$

$\therefore$ $\quad \dfrac{\mathbf{\Delta p_m}}{\mathbf{\Delta p_p}} = \dfrac{\mathbf{\rho_m u_m^2}}{\mathbf{\rho_p u_p^2}}$, As $\rho_m = \rho_p$ and $u_m/u_p = 10$, $\Delta P_m/\Delta P_P = 10^2 =$ **100.**

Case (ii) If air is used in the model, then

$$\frac{u_m}{u_p} = \frac{\mu_m}{\mu_p} \times \frac{d_p}{d_m} \times \frac{\rho_p}{\rho_m}, \quad \frac{\Delta p_m}{\Delta p_p} = \frac{\rho_m}{\rho_p}\left(\frac{\mu_m}{\mu_p} \times \frac{d_p}{d_m} \times \frac{\rho_p}{\rho_m}\right)^2$$

$$= 100\, \frac{\rho_p}{\rho_m}\left(\frac{\mu_m}{\mu_p}\right)^2$$

From data tables at 20^0 C, $\rho_{air} = 1.205$ kg/m^3, $\mu_{air} = 18.14 \times 10^{-6}$ kg/ms,

$\rho_w = 1000$ kg/m^3, $\mu_w = 1.006 \times 10^{-3}$ kg/ms

$\therefore$ $$\frac{\mathbf{\Delta P_m}}{\mathbf{\Delta P_p}} = 100 \times \frac{1000}{1.205} \times \left(\frac{18.14 \times 10^{-6}}{1.006 \times 10^{-3}}\right)^2 = \mathbf{26.98}$$

This illustrates that it may be necessary to use a different fluid in the model as compared to the prototype.

Problem 5.2 *To determine the pressure drop in a square pipe of 1 m side for air flow, a square pipe of 50 mm side was used with water flowing at 3.6 m/s. The pressure drop over a length of 3 m was measured as 940 mm water column. Determine the corresponding* ***flow velocity*** *of air in the larger duct and also the* ***pressure drop*** *over 90 m length. Kinematic viscosity of air = 14.58 × 10^{-6} m^2/s. Density = 1.23 kg/m^3*. Kinematic viscosity of water = 1.18 × 10^{-6} m^2/s

For pipe flow, Reynolds number analogy should be used. Also the drag coefficients will be equal.

For square section hydraulic mean diameter = 4 A/P = $4a^2/4a = a$ (side itself)

$$Re = uD/\nu = 3.6 \times 0.05/1.18 \times 10^{-6} = 152542$$

For air $\qquad 152542 = \dfrac{1 \times u}{14.58 \times 10^{-6}} \qquad \therefore \quad$ **u = 2.224 m/s**

Drag coefficient $F/\rho u^2$ should be the same for both pipes.

$$\frac{F_{air}}{F_w} \frac{\rho_{air} u_{air^2}}{\rho_w u_w^{2}}$$

The pressure drop equals the shear force over the area. For square section, area = a^2, perimeter = $4a$

$$\therefore \qquad \Delta P = \frac{4FL}{a},\ \Delta P_{air} = \frac{4F_{air}L_{air}}{a_{air}},\ \Delta P_{w} = \frac{4F_w L_w}{a_w}$$

Dividing and substituting for F_{air}/F_w

$$\frac{\Delta P_{air}}{\Delta P_w} = \frac{L_{air}}{L_w}\times\frac{a_w}{a_{air}}\times\frac{F_{air}}{F_w} = \frac{L_{air}}{L_w}\times\frac{a_w}{a_{air}}\times\frac{\rho_{air}}{\rho_w}\left(\frac{u_{air}}{u_w}\right)^2$$

$$= \frac{90\times 0.05^2\times 1.23}{1\times 3\times 1000}\left(\frac{2.224}{3.6}\right)^2 = 3.521\times 10^{-5}$$

$$\mathbf{\Delta P_{air}} = 940\times 3.521\times 10^{-5} = \textbf{0.033 mm of water column}$$

Problem 5.3 *Water at 15˚C flowing in a 20 mm pipe becomes turbulent at a velocity of 0.114 m/s. What will be the* ***critical velocity of air at 10⁰C*** *in a similar pipe of 40 mm diameter. Density of air = 1.23kg/m³. Dynamic viscosity of air = 17.7 × 10⁻⁶ kg/ms.*

Density of water = 1000 kg/m³. Dynamic viscosity of water = 1.12 × 10⁻³ kg/ms.

As roughness etc are similar, for pipe flow, reynolds number similarity is to be used.

$$\frac{114\times 0.02\times 1000}{112\ \times 10^{-3}} = \frac{u_{air}\times 0.04\times 1.23}{127\times 10^{-6}}, \quad \therefore \quad \mathbf{u_{air} = 0.732\ m/s}$$

Problem 5.4 *A model of 1/8 geometric scale of a valve is to be designed. The diameter of the prototype is 64 cm and it should control flow rates upto 1m³/s.* ***Determine the flow required for model*** *testing. The valve is to be used with brine in a cooling system at –10⁰C. The kinematic viscosity of brine at the saturated condition is 6.956 × 10⁻⁶ m²/s. For model testing water at 30⁰C is used. Kinematic viscosity is 0.8315 × 10⁻⁶ m²/s*

This is a situation of flow through closed conduits. Reynolds number similarity is required.

$$\frac{u_m d_m}{v_m} = \frac{u_p d_p}{v_p}, \qquad \therefore \quad \frac{u_m}{u_p} = \frac{d_p}{d_m}\times\frac{v_m}{v_p}, \qquad Q_p = \frac{\pi D_p^{\ 2}}{4}u_p$$

$$1 = \pi\times\frac{0.64^2}{4}u_p \qquad \therefore \quad \mathbf{u_p = 3.1085\ m/s}$$

$$\therefore \qquad \mathbf{u_m} = 3.1085\times 8\times 0.8315\times 10^{-6}/6.956\times 10^{-6} = \mathbf{2.9726\ m/s,}$$

$$d_m = d_p/8 = 0.64/8 = 0.08\ \text{m}$$

$$\mathbf{Q_m} = \frac{\pi d_m^{\ 2}}{4}u_m = \frac{\pi}{4}\times 0.08^2\times 2.9726 = \mathbf{0.0149\ m^3/s}$$

If the valve is to be used with water, then the model velocity has to be 8 × 3.1085 m/s. *i.e.* 24.87 m/s, which is rather high.

The pressure drop can also be predicted from the model measurements using

$$\left(\frac{\Delta p}{\rho u^2}\right)_p = \left(\frac{\Delta p}{\rho u^2}\right)_m$$

Problem 5.5 *To predict the drag on an aircraft at a flight speed of 150 m/s, where the condition of air is such that the local speed of sound is 310 m/s, a pressurised low temperature tunnel is used. Density, viscosity and local sonic velocity at tunnel condition are 7.5 kg/m^3, 1.22 × 10^{-5} Ns/m^2 and 290 m/s.* ***Determine the flow velocity and the scale*** *of the model. Assume full dynamic similarity should be maintained. Density and viscosity at the operating conditions are 1.2 kg/m^3 and 1.8 × 10^{-5} Ns/m^2.*

In addition to Reynolds number similarity compressibility effect should be considered. For Mach number similarity,

$$M_p = M_m,\ \frac{u_p}{c_p} = \frac{u_m}{c_m} \quad \therefore \quad u_m = 150 \times 290/310 = \mathbf{140.32\ m/s}$$

For Reynolds number similarity

$$\frac{u_m \rho_m L_m}{\mu_m} = \frac{u_p \rho_p L_p}{\mu_p}$$

$$\frac{\mathbf{L_m}}{\mathbf{L_p}} = \frac{u_p}{u_m} \times \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p} = \frac{150}{140.32} \times \frac{1.2}{7.5} \times \frac{1.8 \times 10^{-5}}{1.22 \times 10^{-5}} = \mathbf{0.252}$$

or about 1/4th scale. When both Match number similarity and Reynolds number similarity should be maintained, generally the size of the model has to be on the higher side Drag force similarity is given by $(F/\rho u^2 L^2)_m = (F/\rho u^2 L^2)_p$

$$\frac{F_m}{F_p} = \frac{\rho_m {u_m}^2 {L_m}^2}{\rho_p {u_p}^2 {L_p}^2} = \frac{7.5}{1.2} \times \frac{140.32^2}{150^2} \times (0.252)^2 = 0.347$$

As the model size is larger, the force ratio is high.

Problem 5.6 *An aircraft fuselage has been designed for speeds of 380 kmph. To estimate power requirements the drag is to be determined. A model of 1/10 size is decided on. In order to reduce the effect of compressibility, the model is proposed to be tested at the same speed in a pressurized tunnel.* ***Estimate the pressure required.*** *If the drag on the model was measured as 100 N, predict the drag on the prototype.*

This is fully immersed flow. Hence Reynolds number similarity is required.

$$\frac{u_m L_m \rho_m}{\mu_m} = \frac{u_p L_p \rho_p}{\mu_p}$$

A viscosity is not affected by pressure and as velocities are equal,

$$L_m \rho_m = L_p \rho_p \quad \therefore \quad \rho_m/\rho_m = L_p/L_m = 10$$

At constant temperature, pressure ratio will be the same as density ratio.

$$\therefore \qquad \mathbf{P_m} = \frac{L_p}{L_m} P_p = \mathbf{10 \times P_p}$$

or 10 times the operating pressure of the aircraft.

The π parameter for drag force, D, gives

$$\frac{D_m}{(1/2)\rho_m u_m{}^2 L_m{}^2} = \frac{D_p}{(1/2)\rho_p u_p{}^2 L_p{}^2} \text{ as } u_m = u_p$$

$\therefore$ $\mathbf{D_p} = D_m\,(\rho_p L_p{}^2/\rho_m L_m{}^2) = 100 \times (1/10)\, 10^2 =$ **1000 or 1 kN**

Problem 5.7 *The performance of an aeroplane to fly at 2400 m height at a speed of 290 kmph is to be evaluated by a 1/8 scale model tested in a pressurised wind tunnel maintaining similarity. The conditions at the flight altitude are temperature = – 1⁰C, pressure = 75 kN/m².*

$\mu = 17.1 \times 10^{-6}$ kg/ms. The test conditions are 2150 kN/m², and 15⁰C.

$\mu = 18.1 \times 10^{-6}$ kg/ms. The drag resistance on the model measured at 18 m/s and 27 m/s. are 4.7N and 9.6N. ***Determine the drag on the prototype.***

At the given flight conditions, Velocity of sound is

$$C = \sqrt{g_0 kRT} = \sqrt{1 \times 1.4 \times 287 \times 272} = 330 \; m/s = 1190 \text{ kmph}$$

Mach number = 290/1190 = 0.24 < 0.3

Hence Reynolds number similarity only need be considered.

Density at test conditions = $2150 \times 10^3/(287 \times 288) = 26.01$ kg/m³

Density at flight conditions = $75 \times 10^3/(287 \times 272) = 0.961$ kg/m³

Equating Reynolds numbers, assuming length L,

Velocity at flight condition = 290000/3600 = 80.56 m/s

$$\frac{80.56 \times L \times 0.961}{17.1 \times 10^{-6}} = u \times \frac{L}{8} \times \frac{26.01}{18.1 \times 10^{-6}} \qquad \therefore \quad \mathbf{u = 25.195 \; m/s}$$

This is also low subsonic. Drag can be obtained using drag coefficient $F/\rho A u^2$

$$\frac{F_m}{\rho_m A_m u_m{}^2} = \frac{F_p}{\rho_p A_p u_p{}^2} \qquad \therefore \quad \frac{F_p}{F_m} = \frac{\rho_p}{\rho_m} \times \left(\frac{u_p}{u_m}\right)^2 \times \frac{A_p}{A_m}$$

$$= \frac{26.01}{0.961} \times \left(\frac{80.56}{25.195}\right)^2 \times 8^2 = 24.165$$

By interpolation using equality of F/u^2, **drag at 25.195 m/s** model speed is obtained as **8.78 N.** $\therefore$ **Drag on prototype =** 8.78 × 24.165 = **212 N**

Problem 5.8 *In a test in a wind tunnel on 1:16 scale model of a bus, at an air speed of 35m/s, the drag on the model was measured as 10.7N, If the width and frontal area of the prototype was 2.44 m and 7.8 m²,* ***estimate the aerodynamic drag*** *force on the bus at 100 kmph. Conditions of air in the wind tunnel are the same as at the operating conditions of the bus. Assume that coefficient of drag remains constant above Reynolds number 10^5.*

$v = 1.006 \times 10^{-6}$ m²/s. Also ***determine the power required.***

The width of the model = 2.44/16 = 0.1525 m.

$$Re = \frac{0.1525 \times 35}{15.06 \times 10^{-6}} = 3.5 \times 10^5, \quad \text{This condition is above } 10^5.$$

Area of the model = 7.8/16²

$$C_D = \frac{F}{(1/2)\rho u^2 A} = \frac{10.7 \times 2 \times 16^2}{1.205 \times 35^2 \times 7.8} = 0.4758$$

Drag force on the prototype at 100 kmph. (27.78 m/s)

$$0.4758 = \frac{F}{(1/2) 1.205 \times 7.8 \times (27.78)^2} \qquad \therefore \quad \mathbf{F = 1725\ N \quad or \quad 1.725\ kN}$$

Power required = 1725 × 27.78 W = 47927 W or **47.927 kW.**

Problem 5.9 *A water tunnel operates with a velocity of 3m/s at the test section and power required was 3.75 kW. If the tunnel is to operate with air,* ***determine for similitude the flow velocity and the power required.***

$$\rho_a = 1.25\ \text{kg/m}^3,\ v_a = 14.8 \times 10^{-6}\ \text{m}^2/\text{s}, \quad v_w = 1.14 \times 10^{-6}\ \text{m}^2/\text{s}$$

In this case Reynolds number similarity is to be maintained. The length dimension is the same.

$$\frac{u_a}{v_a} = \frac{u_w}{v_w}$$

$$\therefore \text{ Velocity of air,} \quad \mathbf{u_a} = \frac{u_w}{v_w} v_a = \frac{3 \times 14.8 \times 10^{-6}}{1.14 \times 10^{-6}} = \mathbf{38.95\ m/s}$$

Power can be determined from drag coefficient, by multiplying and dividing by u as

$F \times u$ power $\qquad \dfrac{F \times u}{\rho A u^2 u} = \dfrac{P}{\rho A u^3} \qquad$ As A is the same,

$$P_{air} = P_w \frac{\rho_{air}}{\rho_w} \frac{u_{air}{}^3}{u_w{}^3} = 3.75 \times \frac{1.28}{1000} \times \left(\frac{38.95}{3}\right)^3 = \mathbf{10.5\ kW}$$

Problem 5.10 *The performance of a torpedo, 1 m diameter and 4 m long is to be predicted for speeds of 10 m/s. If a scale model of 1/25 size is used to predict the performance using a water tunnel,* ***determine the flow velocity*** *required. The ratio of density between sea water and fresh water is 1.02 and the viscosity ratio is 1.05. Also* ***determine the value of Reynolds number,*** *if the density of water was 1000 kg/m³ and kinematic viscosity was 0.832 × 10⁻⁶ m²/s.*

This is a fully submerged flow. Hence Reynolds number similarity should be maintained in the test. *i.e.* $\dfrac{D_p u_p \rho_p}{\mu_p} = \dfrac{D_m u_m \rho_m}{\mu_m}$,

$$u_m = u_p \times \frac{D_p}{D_m} \times \frac{\rho_p}{\rho_m} \times \frac{\mu_m}{\mu_p} = 10 \times 25 \times 1.02/1.05 = 242.85\ \text{m/s}$$

This is a very high speed generally not achievable in water tunnel.

$$Re = D_m\, u_m\, \rho_m/\mu_m = \frac{1}{25} \times \frac{242.85 \times 1000}{1000 \times 0.832 \times 10^{-6}} = 11.67 \times 10^6$$

For values of $Re > 10^5$ the coefficient of drag remains constant. Hence strict Reynolds number similarity need not be insisted on beyond such value.

In this case for example, velocity around 2.5 m/s may be used for the test which corresponds to $Re = 1.2 \times 10^5$.

Problem 5.11 *A 1/6 scale model of a submarine is tested in a wind tunnel using air of density 28.5 kg/m³ and viscosity 18.39 × 10⁻⁶ kg/ms at a speed of 36.6 m/s.* ***Calculate the corresponding speed and drag*** *of the prototype when submerged in sea water with density 1025 kg/m³ and viscosity 1.637 × 10⁻³ kg/ms if the model resistance was 67 N.*

Reynolds number similarity should be considered in this case. Let L be the length of the prototype.

$$\frac{36.6 \times L \times 28.5}{6 \times 18.39 \times 10^{-6}} = \frac{u_p \times L \times 1025}{1.637 \times 10^{-3}} \qquad \therefore\ \mathbf{u_p = 15.2\ m/s}$$

Using drag coefficient

$$\frac{F_m}{\rho_m u_m{}^2 L_m{}^2} = \frac{F_p}{\rho_p u_p{}^2 L_p{}^2}$$

$$\therefore \qquad \mathbf{F_p} = F_m \times \frac{\rho_p}{\rho_m}\left(\frac{u_p}{u_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 = 67 \times \frac{1025}{28.5}\left(\frac{15.2}{36.6}\right)^2 (6)^2 = \mathbf{14961\ N}$$

Problem 5.12 *A sonar transducer in the shape of a sphere of 200 mm diameter is used in a boat to be towed at 2.6 m/s in water at 20°C. To determine the drag on the transducer a model of 100 mm diameter is tested in a wind tunnel, the air being at 20° C. The drag force is measured as 15 N. Determine the* ***speed of air for the test. Estimate the drag*** *on the prototype.*

As it is fully immersed type of flow, Reynolds number similarity should be maintained.

The density and kinematic viscosity values are :

$$\rho_{air} = 1.205\ \text{kg/m}^3, \quad v_{air} = 15.06 \times 10^{-6}\ \ \text{m}^2/\text{s}$$
$$\rho_w = 1000\ \text{kg/m}^3, \quad v_w = 1.006 \times 10^{-6}\ \ \text{m}^2/\text{s}$$

$$\frac{u_m D_m}{v_m} = \frac{u_p D_p}{v_p}$$

$$\mathbf{u}_m = u_p \frac{D_p}{D_m}\frac{v_m}{v_p} = 2.6 \times \frac{200}{100} \times \frac{15.06 \times 10^{-6}}{1.006 \times 10^{-6}} = \mathbf{77.85\ m/s}$$

Mach number will be about 0.25. Hence compressibility effect will be negligible.

The coefficient of drag should be same for this condition. As $A \propto D^2$

$$\frac{F_m}{\rho_m u_m{}^2 D_m{}^2} = \frac{F_p}{\rho_p u_p{}^2 D_p{}^2},$$

$$\therefore \qquad \mathbf{F_p} = 15 \times \frac{1000}{1.205} \times \left[\frac{2.6 \times 200}{77.85 \times 100}\right]^2 = \mathbf{55.54\ N}$$

Problem 5.13 *In order to predict the flow conditions after the turbine outlet (tail race) of a hydroelectric plant delivering 2400 m³/s, a model of 1/75 scale is proposed.* ***Determine the flow rate required.***

This is a free surface flow. Hence Froude number similarity is to be maintained.

$$Fr_m = Fr_p \qquad \text{or} \qquad \frac{u_m}{\sqrt{l_m}} = \frac{u_p}{\sqrt{l_p}} \qquad \text{or} \qquad \frac{u_m}{u_p} = \sqrt{\frac{l_m}{l_p}}$$

As flow ($Q = Au$) depends on area which varies as L^2

$$\therefore \qquad \frac{Q_m}{Q_p} = \frac{A_m u_m}{A_p u_p} = \frac{{L_m}^2}{{L_p}^2}\sqrt{\frac{L_m}{L_p}} = \left(\frac{L_m}{L_p}\right)^{2.5}$$

$$\therefore \qquad \mathbf{Q_m} = 2400\left(\frac{1}{75}\right)^{2.5} = \mathbf{0.04927\ m^3/s}$$

Problem 5.14 *The total drag on a ship having a wetted hull area of 2500 m^2 is to be estimated. The ship is to travel at a speed of 12 m/s. A model 1/40 scale when tested at corresponding speed gave a total resistance of 32 N. From other tests the frictional resistance to the model was found to follow the law $F_{sm} = 3.7\ u^{1.95}$ N/m² of wetted area. For the prototype the law is estimated to follow $F_{sp} = 2.9\ u^{1.8}$ N/m² of wetted area* ***Determine the expected total resistance.***

The total resistance to ships movement is made up of (*i*) wave resistance and (*ii*) frictional drag. For wave resistance study Froude number similarity should be maintained. For frictional resistance Reynolds number similarity should be maintained. But it is not possible to maintain these similarities simultaneously. In the case of ships the wave resistance is more difficult to predict. Hence Froude number similarity is used to estimate wave resistance. Frictional drag is estimated by separate tests. From the Froude number similarity,

$$\mathbf{u_m} = u_p\sqrt{\frac{l_m}{l_p}} = 12/40^{0.5} = \mathbf{1.897\ m/s}$$

The **skin friction drag** for the model is calculated using this velocity.

$$F_{sm} = 3.7 \times 1.897^{1.95} \times A_m \quad \text{as} \quad A_m = 2500/40^2$$
$$= 3.7 \times 1.897^{1.95} \times 2500/40^2 = \mathbf{20.16\ N}$$

Wave drag on the model = 32 – 20.16 = **11.84 N**

The wave drag is calculated using $(F/\rho\ u^2L^2)_m = (F/\rho\ u^2L^2)_p$

Noting that sea water is denser with $\rho = 1025$ kg/m³

$$F_{wp} = F_{wm}\,\frac{\rho_p}{\rho_m}\left(\frac{L_p}{L_m}\right)^2\left(\frac{u_p}{u_m}\right)^2 = 11.84 \times \frac{1025}{1000}\,(40)^2\left(\frac{12}{1.897}\right)^2 = 774.38 \times 10^3\ \text{N}$$

Skin friction drag for the prototype

$\mathbf{F_{sp}} = 2.9\ {u_p}^{1.8} \times A_p = 2.9 \times 12^{1.8} \times 2500 = 635.13 \times 10^3$ N

$\therefore$ Total resistance = 1.41×10^6 N or **1.41 MN**

Problem 5.15 *A scale model of a ship of 1/30 size is to be towed through water. The ship is 135 m long. For similarity* ***determine the speed with which the model should be towed.*** *The ship is to travel at 30 kmph.*

Froude number similarity is to be maintained.

$$\frac{u_m}{\sqrt{gL_m}} = \frac{u_p}{\sqrt{gL_p}} \quad \therefore \quad \boldsymbol{u_m} = u_p\sqrt{\frac{L_m}{L_p}} = \frac{30 \times 1000}{3600} \times \frac{1}{\sqrt{30}} = \mathbf{1.52\ m/s}$$

Problem 5.16 *The wave resistance of a ship when travelling at 12.5 m/s is estimated by test on 1/40 scale model. The resistance measured in fresh water was 16 N.* ***Determine the speed of the model and the wave resistance of the prototype*** *in sea water. The density of sea water = 1025 kg/m³.*

Froude number similarity is to be maintained.

$$\therefore \qquad \mathbf{u_m} = u_p \sqrt{\frac{L_m}{L_p}} = 12.5 \sqrt{\frac{1}{40}} = \mathbf{1.976\ m/s}$$

The wave resistance is found to vary as given below.

$$\frac{F_m}{\rho_m {u_m}^2 {L_m}^2} = \frac{F_p}{\rho_p {u_p}^2 {L_p}^2}$$

$$\therefore \qquad \mathbf{F_p} = F_m \times \frac{\rho_p}{\rho_m}\left(\frac{u_p}{u_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 = 16 \times \frac{1025}{1000}\left(\frac{12.5}{1.976}\right)^2 (40)^2$$

$$= \mathbf{1049.6 \times 10^3\ N} \quad \text{or} \quad \mathbf{1050\ kN}$$

Problem 5.17 *Vortex shedding at the rear of a structure of a given section can create harmful periodic vibration. To predict the shedding frequency, a smaller model is to be tested in a water tunnel. The air speed is expected to be about 65 kmph. If the geometric scale is 1:6 and if the water temperature is 20° C determine the speed to be used in the tunnel. Consider air temperature as 40° C. If the shedding frequency of the model was 60 Hz* ***determine the shedding frequency*** *of the prototype. The dimension of the structure are diameter = 0.12 m, height = 0.36 m.*

The frequency of vortex shedding can be related by the equation

$$\omega = F(d, h, u, \rho, \mu)$$

Dimensional analysis leads to the π terms relation, (refer Chapter 8)

$$\frac{\omega D}{u} = f\left(\frac{D}{H}, \frac{\rho u D}{\mu}\right)$$

The model dimension can be determined as

$$D_m = 1/6\ D_p = 0.12/0.6 = 0.02\ \text{m},\ H_m = 1/6\ H_p = 0.36/0.6 = 0.06\ m$$

$$\therefore \qquad \frac{D}{H} = \frac{0.02}{0.06} = \frac{1}{3}, \qquad \text{Reynolds similarity requires}$$

$$\frac{\rho_m u_m D_m}{\mu_m} = \frac{\rho_p u_p D_p}{\mu_p} \qquad \therefore \quad u_m = u_p \frac{\rho_p}{\rho_m}\frac{\mu_m}{\mu_p}\frac{D_p}{D_m}$$

The property values of air and water at the given temperatures are,

$$\rho_p = 1.128\ \text{kg/m}^3, \quad \mu_p = 19.12 \times 10^{-6}\ \text{kg/ms}$$

$$\rho_m = 1000\ \text{kg/m}^3, \quad \mu_p = 1.006 \times 10^{-6}\ \text{kg/ms}$$

$$u_p = 65 \times 1000/3600 = 18.056\ \text{m/s}$$

$$\therefore \qquad \mathbf{u_m} = 18.056 \times \frac{1.128}{1000} \times \frac{1.006 \times 10^{-3}}{19.12 \times 10^{-6}}\,\frac{6}{1} = \mathbf{6.43\ m/s}$$

Vortex shedding frequency is determined. Using the third π parameter,

$$\frac{\omega_m D_m}{u_m} = \frac{\omega_p D_p}{u_p}$$

$$\therefore \qquad \boldsymbol{\omega_p} = \frac{u_p}{u_m}\,\frac{D_p}{D_m}\;\omega_m = \frac{18.06}{6.43} \times \frac{1}{6} \times 60 = \mathbf{28.08\ Hz.}$$

The drag also can be predicted from the model. The drag for unit length can be expressed in the dimensionless from as $D/d\rho u^2$ where D is the drag and d is the diameter. Thus

$$\frac{D_p}{d_p \rho_p {u_p}^2} = \frac{D_m}{d_m \rho_m {u_m}^2} \quad \therefore \quad D_p = D_m \cdot \frac{d_p}{d_m} \cdot \frac{\rho_p}{\rho_m} \cdot \frac{{u_p}^2}{{u_m}^2}$$

Problem 5.18 *In order to determine the drag on supporting columns (of a bridge) of* 0.3 *m diameter, due to water flowing at a speed of 14.5 km/hr, a column of 0.25 m diameter was tested with air flow. The resistance was measured as 227 N/m, under similar conditions of flow.* ***Determine the force on the bridge*** *column per m length.* $v_{air} = 1.48 \times 10^{-5}\ m^2/s$. $v_w = 1.31 \times 10^{-6}\ m^2/s, \rho_a = 1.23\ kg/m^3$

Similarity requires equal Reynolds numbers

Velocity of flow of water = 14.5 × 1000/3600 = 4.028 m/s

$$\frac{4.028 \times 0.3}{1.31 \times 10^{-6}} = \frac{u_a \times 0.25}{1.48 \times 10^{-5}}$$

∴ Velocity of air, u_a = 54.61 m/s

The force can be obtained by the dimensional parameter (drag coefficient)

$F/\rho A u^2$, here $A = 1 \times D$

∴ The parameter in this case for force is

$$\frac{F}{\rho D u^2} \quad \frac{F_w}{\rho_w D_w {u_w}^2} = \frac{F_a}{\rho_a D_a {u_a}^2} \quad \text{or}$$

$$\mathbf{F_w} = F_a \left(\frac{\rho_w}{\rho_a}\right)\left(\frac{D_w}{D_a}\right)\left(\frac{u_w}{u_a}\right)^2 = 227 \times \frac{1000}{1.23} \times \frac{0.3}{0.25} \times \left(\frac{4.028}{54.61}\right)^2 = \mathbf{1205\ N}$$

Problem 5.19 *To ascertain the flow characteristics of the spillway of a dam, 1/20 geometric scale model is to be used. The spillway is 40 m long and carries 300 m³/s at flood condition.* ***Determine the flow rate required*** *to test the model. Also* ***determine the time scale*** *for the model. Viscous and surface tension effects may be neglected.*

This situation is open surface flow. Froude number similarity is required.

$$\frac{u_m}{(gL_m)^{0.5}} = \frac{u_p}{(gL_p)^{0.5}} \quad \text{or} \quad \therefore \frac{u_m}{u_p} = \left(\frac{L_m}{L_p}\right)^{0.5}$$

As $$Q = uA = uL^2,\ Q_m = u_m L_m^2,\ Q_p = u_p L_p^2,\ \frac{Q_m}{Q_p} = \frac{u_m L_m^2}{u_p L_p^2}$$

$$\therefore \qquad \mathbf{Q_m} = Q_p \frac{u_m}{u_p} \frac{L_m^{\ 2}}{L_p^{\ 2}} = Q_p \left(\frac{L_m}{L_p}\right)^{0.5} \left(\frac{L_m}{L_p}\right)^{2} = Q_p \left(\frac{L_m}{L_p}\right)^{2.5}$$

$$= 300 \left(\frac{1}{20}\right)^{2.5} = \mathbf{0.168\ m^3/s}$$

Time scale can be determined from velocities, as velocity = length/time.

$$\frac{u_m}{u_p} = \frac{L_m}{L_p} \frac{t_p}{t_m} \quad \therefore \quad \frac{\mathbf{t_m}}{\mathbf{t_p}} = \frac{L_m}{L_p} \frac{u_p}{u_m} = \frac{L_m}{L_p} \left(\frac{L_p}{L_m}\right)^{0.5} = \left(\frac{L_m}{L_p}\right)^{0.5} = \left(\frac{1}{20}\right)^{0.5} = \mathbf{0.2236}$$

Problem 5.20 *A fan when tested at ground level with air density of 1.3 kg/m³, running at 990 rpm was found to deliver 1.41 m³/s at a pressure of 141 N/m². This is to work at a place where the air density is 0.92 kg/m³, the speed being 1400 rpm.*

Determine the volume delivered and the pressure rise.

For similarity condition the flow coefficient Q/ND^3 should be equal.

As D is the same,

$$\frac{Q_1}{N_1} = \frac{Q_2}{N_2} \qquad \text{or} \qquad \mathbf{Q_2} = Q_1 \frac{N_2}{N_1} = 1.41 \times \frac{1400}{990} = \mathbf{2\ m^3/s}$$

The head coefficient $H/\rho N^2 D^2$ is used to determine the pressure rise.

$$\mathbf{\Delta P_2} = \Delta P_1 \frac{\rho_2 N_2^{\ 2}}{\rho_1 N_1^{\ 2}} = 141 \times \frac{0.92}{1.3} \times \left(\frac{1400}{990}\right)^2 = \mathbf{199.55\ N/m^2}$$

Problem 5.21 *A centrifugal pump with dimensional specific speed (SI) of 2300 running at 1170 rpm delivers 70 m³/hr. The impeller diameter is 0.2 m. Determine* ***the flow, head and power if*** *the pump runs at 1750 rpm . Also* ***calculate the specific speed*** *at this condition.*

The head developed and the power at test conditions are determined first. (At 1170 rpm).

$$N_s = N\sqrt{Q}/H^{3/4} = 1170\sqrt{70}/H^{3/4} = 2300 \quad \therefore \mathbf{H = 6.9\ m}$$

$$\textbf{Power} = \text{mg H} = 9.81 \times 70000 \times 6.9/3600 = \mathbf{1316\ W}$$

When operating at 1750 rpm, using flow coefficient Q/ND^3, as D is the same

$$\mathbf{Q_2} = 70 \left(\frac{1750}{1170}\right) = \mathbf{104.7\ m^3/hr}$$

Using head coefficient, H/N^2D^2, $\mathbf{H_2} = H_1\ (N_1/N_2)^2 = 6.9 \times (1750/1170)^2 = \mathbf{15.44\ m}$

Using power coefficient : P/rN^3D^5,

$$\mathbf{P_2} = P_1 \times \left[\frac{N_2}{N_1}\right]^3 = 1316 \times \left[\frac{1750}{1170}\right]^3 = \mathbf{4404\ W}$$

Specific speed for the model

$$N_s = N\sqrt{Q}/H^{3/4} = 1750\sqrt{104.7}/(15.44)^{3/4} = 2300$$

Note: Specfic speeds are the same.

Problem 5.22 *A pump running at 1450 rpm with impeller diameter of 20 cm is geometrically similar to a pump with 30 cm impeller diameter running at 950 rpm. The discharge of the larger pump at the maximum efficiency was 200 litres/s at a total head of 25m.* ***Determine the discharge*** *and head of the smaller pump at the maximum efficiency conditions. Also determine the* ***ratio of power*** *required.*

The *PI* terms of interest are the head coefficient, power coefficient scale and $Q/\omega D^3$ called flow coefficient ($\omega \propto N$) (Refer chapter 8, Problem 8.16).

Considering flow coefficient, denoting the larger machines as 1 and the smaller as 2,

$$\frac{Q_1}{\omega_1 D_1^{\,3}} = \frac{Q_2}{\omega_2 D_2^{\,3}} \quad \therefore \quad \mathbf{Q_2} = Q_1 \frac{\omega_2}{\omega_1}\frac{D_2^{\,3}}{D_1^{\,3}} = 200 \times \frac{1450}{950}\left(\frac{20}{30}\right)^3 = \mathbf{90.45\ l/s}$$

Considering head coefficient, (g being common)

$$\frac{gh_1}{\omega_1^{\,2} D_1^{\,2}} = \frac{gh_2}{\omega_2^{\,2} D_2^{\,2}} \quad \therefore \ h_2 = h_1\left(\frac{\omega_2}{\omega_1}\right)^2\left(\frac{D_2}{D_1}\right)^2$$

$$\therefore \qquad \mathbf{h_2} = 25 \times \left[\frac{1450}{950}\right]^2\left(\frac{20}{30}\right)^2 = \mathbf{25.885\ m}$$

Consider power coefficient $\dfrac{P_1}{\rho_1 \omega_1^{\,3} D_1^{\,5}} = \dfrac{P_1}{\rho_1 \omega_1^{\,3} D_1^{\,5}}$, as $\rho_1 = \rho_2$,

$$\frac{\mathbf{P_2}}{\mathbf{P_1}} = \left(\frac{\omega_2}{\omega_1}\right)^3\left(\frac{D_2}{D_1}\right)^5 = \left(\frac{1450}{950}\right)^3\left(\frac{20}{30}\right)^5 = \mathbf{0.468}$$

As efficiencies should be the same, $Q_1\,\rho_1\,h_1 = Q_2\,\rho_2\,h_2$, with $\rho_1 = \rho_2$

$$0.200 \times 25 = 0.09045 \times 25.885/0.468, \ 5.00 = 5.00 \text{ (checks)}$$

Specific speed $= N\sqrt{Q}/H^{3/4} = 1450\sqrt{0.09045}\,/\,25.885^{3/4} = 38$ (dimensional)

For larger pump, specific speed = $950\sqrt{0.2}/25^{3/4} = 38$, checks.

Problem 5.23 *A V notch is to be used with utectic calcium chloride solution at 30°C. Density = 1282 kg/m^3, v_e = 2.267 × 10^{-6} m^2/s. The flow rate has to be found for various heads. Water was used for the test at 20°C. Density = 1000 kg/m^3, v_w = 1.006 × 10^{-6} m^2/s. Neglecting the effect of surface tension,* ***determine the ratio of corresponding heads*** *and* ***mass flow rates of water and*** *the solution at the corresponding heads.*

Dimensional analysis shows (Neglecting surface tension effects), Q being volume flow rate that for similarity the following parameters should be equal. (suffix c refers to the solution properties) (Refer chapter 8, Problem 8.13).

$$\frac{Q}{g^{1/2} h^{5/2}} = f\left[\frac{g^{1/2} h^{3/2}}{v}, \theta\right]$$

$$\therefore \qquad \frac{h_w^{\,3/2}}{v_w} = \frac{h_c^{\,3/2}}{v_c}$$

$$\therefore \quad \frac{\mathbf{h_c}}{\mathbf{h_w}} = \left(\frac{v_c}{v_w}\right)^{2/3} = \left[\frac{2.267 \times 10^{-6}}{1.006 \times 10^{-6}}\right]^{2/3} = \mathbf{1.71884}$$

$$\frac{Q_c}{h_c^{\,5/2}} = \frac{Q_w}{h_w^{\,5/2}}$$

$$\therefore \quad \frac{\mathbf{Q_c}}{\mathbf{Q_w}} = \left(\frac{h_c}{h_w}\right)^{5/2} = (1.71884)^{5/2} = \mathbf{3.873}$$

Ratio of mass flow rates = 3.873 × (1282/1000) = **4.97**

Problem 5.24 *The discharge Q through an orifice is found to depend on the parameter* $\rho D\sqrt{gH}/\mu$, *when surface tension effect is neglected. Determine the ratio of flow rates of water and refrigerant 12 at 20°C under the same head.* ***What should be the ratio of heads for the*** *same flow rate.* $\mu_R = 2.7 \times 10^{-4}$ *kg/ms.,* $\mu_w = 1.006 \times 10^{-3}$ *kg/ms. Density of refrigerent = 923 kg/m*^{3}.

$$Q_w \propto \rho_w D\sqrt{gH}/\mu_w \quad \text{and} \quad Q_R \propto \rho_R D\sqrt{gH}/\mu_r, \text{ Dividing}$$

$$\frac{\mathbf{Q_R}}{\mathbf{Q_w}} = \frac{\rho_R}{\rho_w} \times \frac{\mu_R}{\mu_w} = \frac{923}{1000} \times \frac{1.006 \times 10^{-3}}{2.7 \times 10^{-4}} = \mathbf{3.44}$$

For the same flow rate

$$\frac{\rho_w D\sqrt{gH_w}}{\mu_w} = \frac{\rho_R D\sqrt{gH_R}}{\mu_R}$$

$$\therefore \quad \frac{\mathbf{H_w}}{\mathbf{H_R}} = \left(\frac{\mu_w}{\mu_R} \times \frac{\rho_R}{\rho_w}\right)^2 = \left(\frac{1.006 \times 10^{-3}}{2.7 \times 10^{-4}} \times \frac{923}{1000}\right)^2 = \mathbf{11.827}$$

OBJECTIVE QUESTIONS

O Q. 9.1 Fill in the blanks.

1. The representation of a physical system used to predict the behaviour of the system is called __________.
2. The system whose behaviour is predicted by the model is called __________.
3. Models are generally __________ in size compared to prototype.
4. When the prototype is very small __________ model is used.
5. Models may also be used to predict the effect of __________ to an existing system.
6. Dimensionless parameters provide __________ conditions for model testing.
7. For geometric similarity ratio of __________ should be equal.
8. For dynamic similarity ratio of __________ should be equal.
9. If stream lines are similar between model and prototype it is called __________ similarity.
10. When geometric and dynamic similarities exist then automatically __________ will exist.

Chapter-6 Boundary Layer Theory and Flow Over Surfaces

6.0 INTRODUCTION

Ideal inviscid fluids do not exert any force on the surfaces over which they flow. Real fluids have viscosity. When these fluids flow over surfaces, "no slip condition" prevails. The layer near the surface has to have the same velocity as the surface. If the surface is at rest, then this layer comes to rest. The adjacent layer is retarded to a lesser extent and this proceeds to layers more removed from the surface at rest. A velocity gradient forms leading to shear force being exerted over the layers. The velocity gradient is steepest at the interface and the shear is also highest at the interface. Work is to be done to overcome the force. The equations for the analysis of the complete flow field has been formulated by Navier and Stokes. But solutions for these equations for practical boundary conditions were not available. For a long time empirical equations based on experimental results were used in designs.

The development of boundary layer theory enabled the analysis of such flows to be fairly easy. The theory was proposed by Ludwig Prandtl in 1904. He observed that **in the case of real fluids velocity gradient existed only in a thin layer near the surface. This layer was named as boundary layer**. Beyond this layer the effect of viscosity was found negligible. This was supported by measurement of velocity. The flow field now can be divided into two regions, one in which velocity gradient and shear existed and another where viscous effects are negligible. This region can be dealt with as flow of inviscid fluid or ideal fluid. In the study of flow over immersed bodies like aircraft wings the analysis can be limited to the boundary layer, instead of the field extending to long distances for the determination of forces exerted on the surface by the fluid flowing over it.

6.1 BOUNDARY LAYER THICKNESS

In the solution of the basic equations describing the flow namely continuity and momentum equations of the boundary layer, one boundary is provided by the solid surface. The need for the other boundary is met by edge of the boundary layer determined by the thickness. The determination of the velocity variation along the layer enables the determination of velocity gradient. This is made possible by these two boundary conditions. Once the velocity gradient at the surface is determined, the shear stress can be determined using the equation

$$\tau = \mu \frac{du}{dy} \tag{6.1.1}$$

This leads to the determination of resistance due to the flow.

6.1.1 Flow Over Flat Plate

The simplest situation that can be analyzed is the flow over a flat plate placed parallel to uniform flow velocity in a large flow field. The layer near the surface is retarded to rest or zero. velocity. The next layer is retarded to a lower extent. This proceeds farther till the velocity equals the free stream velocity. As the distance for this condition is difficult to determine, **the boundary layer thickness is arbitrarily defined as the distance from the surface where the velocity is 0.99 times the free stream velocity.**

There are two approaches for the analysis of the problem.

1. Exact method : Solution of the differential equations describing the flow using the boundary conditions. It is found that this method can be easily applied only to simple geometries.

2. Approximate method : Formulation of integral equations describing the flow and solving them using an assumed velocity variation satisfying the boundary conditions. This method is more versatile and results in easier solution of problems. The difference between the results obtained by the exact method and by the integral method is found to be within acceptable limits.

At present several computer softwares are available to solve almost any type of boundary, and the learner should become familiar with such softwares if he is to be current.

6.1.2 Continuity Equation

The flow of fluid over a flat plate in a large flow field is shown in Fig. 10.1.1. The flow over the top surface alone is shown in the figure.

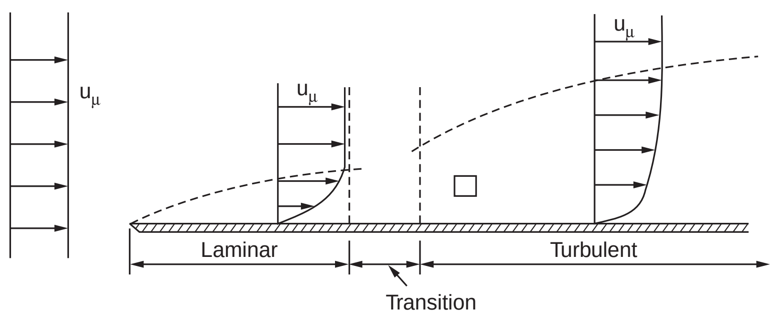


Figure 6.1.1 Formation of boundary layer over flat plate

The velocity is uniform in the flow field having a value of u_∞. Boundary layer begins to form from the leading edge and increases in thickness as the flow proceeds. This is because the viscosity effect is felt at layers more and more removed from the surface. At the earlier stages the flow is regular and layers keep their position and there is no macroscopic mixing between layers. **Momentum transfer resulting in the retarding force is by molecular diffusion**

between layers. This type of flow is called laminar flow and analysis of such flow is somewhat simpler. **Viscous effects prevail over inertial effects in such a layer**. Viscous forces maintain orderly flow. **As flow proceeds farther, inertial effects begin to prevail over viscous forces resulting in macroscopic mixing between layers. This type of flow is called turbulent flow**. Higher rates of momentum transfer takes place in such a flow. For the formulation of the differential equations an element of size $dx \times dy \times 1$ is considered.

An enlarged sectional view of the element is shown in Fig. 6.1.2.

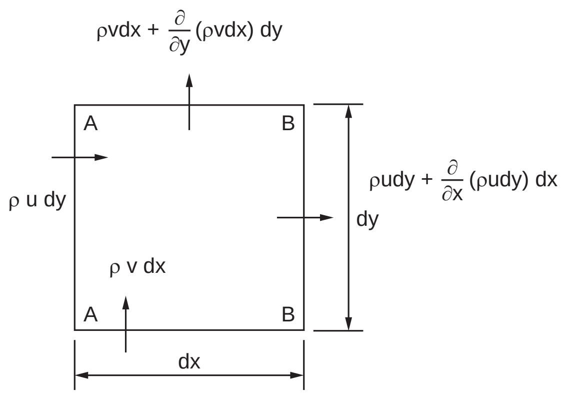


Figure 6.1.2 Enlarged view of element in the boundary layer

The assumptions are (*i*) flow is incompressible or density remains constant, (*ii*) flow is steady, (*iii*) there is no pressure gradient in the boundary layer.

Continuity equation is obtained using the principle of conservation of mass. Under steady flow conditions the net mass flow across the element should be zero. Under unsteady conditions, the net mass flow should equal the change of mass in the elemental volume considered. The values of velocities are indicated in the figure. The density of the fluid is ρ. Unit time and unit Z distance are assumed. **Time is not indicated in the equations.**

Flow in across face AA, $\rho u dy \times 1 = \rho u dy$

Flow out across face BB, $\rho u dy + \dfrac{\partial}{\partial x}(\rho u dy)\, dx$

Net flow in the x direction $= \dfrac{\partial(\rho u)}{\partial x}\, dxdy$

Similarly the net flow in the y direction is given by $\dfrac{\partial(\rho v)}{\partial x}\, dxdy$

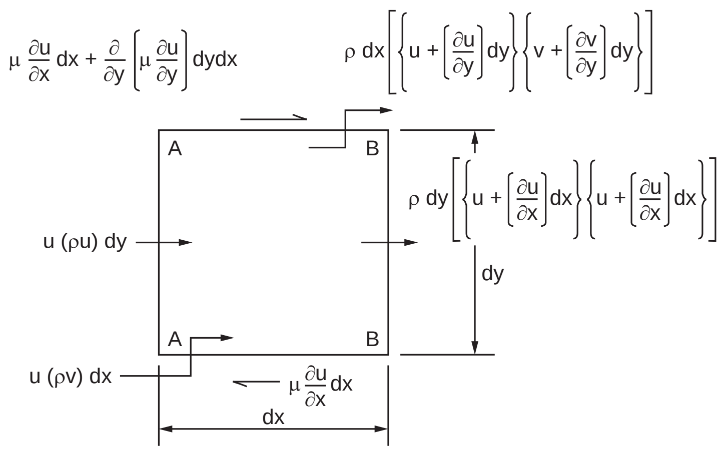
Under steady conditions the sum is zero. Also for incompressible flow density is constant. Hence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6.1.2)$$

This is known as continuity equation for steady incompressible flow. If u decreases, $\dfrac{\partial u}{\partial x}$ is – ve and so $\dfrac{\partial v}{\partial y}$ should be positive. The algebraic sum of x and y directional flows is zero.

6.1.3 Momentum Equation

The equation is based on Newton's second law of motion. The net force on the surface of the element should equal the rate of change of momentum of the fluid flowing through the element. Here x directional forces are considered with reference to the element shown in Fig. 10.1.3. The flows are indicated on the figure unit time and unit Z distance are assumed. The density of the fluid is ρ



Consider the momentum flow in the x direction :

Across AA momentum flow = $u\,(\rho u)\,dy$

Across BB momentum flow = $u\,(\rho u)\,dy + \dfrac{\partial}{\partial x}\{u(\rho u)\,dy\}\,dx$

Taking the difference, the net flow is (as ρ is constant) (u^2 is written as $u \times u$)

$$\frac{\partial}{\partial x}[u(\rho u)dy]dx = \rho\, dxdy\left[u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\right]$$

Considering the flow in the y direction, the net x directional momentum flow is

$$\frac{\partial}{\partial y}[u(\rho v)dy]dx = \rho\, dxdy\left[u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y}\right]$$

Summing up, the net momentum flow is

$$\rho\, dxdy\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \left\{u\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]\right\}\right]$$

From continuity equation, the second set in the above equation is zero. Hence net x directional momentum flow is

$$\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]\rho\, dxdy$$

It was assumed that no body forces or pressure forces are present. Only surface forces due to viscosity is considered.

At the bottom surface shear $= dx\,\mu\,\dfrac{\partial u}{\partial y}$

At the top surface shear $= dx\,\mu\,\dfrac{\partial u}{\partial y} + \dfrac{\partial}{\partial y}\left[\mu\dfrac{\partial u}{\partial y}dx\right]dy$

The net shear on the element is $\mu\dfrac{\partial^2 u}{\partial y^2}dxdy$, noting $v = \mu/\rho$

Equating, and simplifying, $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = v\dfrac{\partial^2 u}{\partial y^2}$

This is known as momentum equation for the boundary layer. v is also called as momentum diffusivity. In case of pressure gradient along the flow $-\dfrac{1}{\rho}\dfrac{\partial P}{\partial x}$ has to added on the RHS.

6.1.4 Solution for Velocity Profile

The continuity and momentum equations should be simultaneously solved to obtain the velocity profile. The boundary conditions are

(i) at $y = 0$, $u = 0$, (ii) at $y = \delta$, $u = u_\infty$, $\dfrac{\partial u}{\partial y} = 0$

The solution for these equations was obtained by Blasius in 1908 first by converting the partial differential equation into a third order ordinary differential equation and then using numerical method.

The two new vaiables introduced were

$$\eta = y\sqrt{\frac{u_\infty}{xv}} \quad \text{and } f(\eta) = \psi/\sqrt{vxu_\infty}$$

where ψ is the stream function giving

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{6.1.5}$$

The resulting ordinary differential equation is

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0 \tag{6.1.6}$$

the boundary conditions with the new variables are

at $y = 0$, $\eta = 0$ and $\dfrac{\partial f}{\partial \eta} = 0$, at $y = \infty$, $\eta = \infty$ and $\dfrac{\partial f}{\partial \eta} = 1$

The results where plotted with u/u_∞ as the dependent vairable and $y\sqrt{\dfrac{u_\infty}{vx}}$ or (η) as the independent variable resulting in a plot as shown in Fig. 10.1.4.

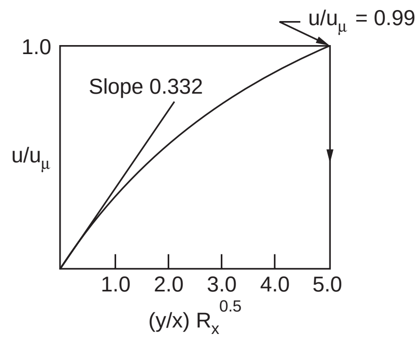


Figure 6.1.4 Velocity distribution in boundary layer

The value of $y\sqrt{\dfrac{u_\infty}{vx}}$ where u/u_∞ = 0.99 is found to be 5. This y value is taken as the boundary layer thickness δ as per the definition of thickness of boundary layer.

i.e. $$\delta\sqrt{u_\infty / vx} = 5, \quad \text{or} \quad \delta = \frac{5}{\sqrt{u_\infty / vx}} = \frac{5x}{\sqrt{u_\infty x / v}} \frac{5x}{\sqrt{Re_x}} = 5x\, Re_x^{-0.5}$$

This equation was more precisely solved in 1983 by Howarth. The significance of Reynolds number has already been explained under dimensional analysis as the ratio of inertia force to viscous force. Velocity gradient at the surface is of greater importance because it decides the shear on the surface at $y = 0$

$$\tau_w = \mu\,\frac{\partial u}{\partial y} \text{ equals the value of } \mu u_\infty \sqrt{u_\infty / vx}\,\frac{d^2 f}{d\eta^2}, \text{ at } \eta = 0$$

From the solution, at $\eta = 0$, the value of $\dfrac{d^2 f}{d\eta^2}$, is obtained as 0.332

Substituting this value and replacing v by μ/ρ and simplifying

$$\tau_w = 0.332\ \rho u_x^{\ 2}/\sqrt{Re} \qquad (6.1.8)^\infty$$

Defining skin friction coefficient, C_{fx}, as $\tau_w/(1/2)\rho u_\infty^{\ 2}$, we obtain

$$C_{fx} = 0.664\ Re_x^{-0.5} \qquad (6.1.9)$$

The average value over length L can be obtained by using

$$C_f = \frac{1}{L}\int_0^L C_{fx}\, dx = 1.328\ Re_L^{-0.5}$$

Not that these results are obtained for laminar flow over flat plate for $Re < 5 \times 10^5$.

Example 6.1. ***Air*** *at 30° C flows over a flat plate at a free stream velocity of 5m/s.* ***Determine the boundary layer thickness*** *at distances 0.2 m, 0.5 m and 0.8 m. Also determine the skin friction coefficients, both local and average, at these locations.*

The property values for air at 30 °C are obtained from tables. $\rho = 1.165$ kg/m^3,

$$v = 16 \times 10^{-6}\ \text{m}^2/\text{s},\ \mu = 18.63 \times 10^{-6}\ \text{kg/ms}.$$

$$\delta = 5x\, Re_x^{-0.5},\ C_{fx} = 0.664\, Re_x^{-0.5},\ C_{fL} = 1.328\, Re_L^{-0.5}$$

Consider 0.5 m, $$Re_x = \frac{ux}{v} = \frac{5 \times 0.5}{16 \times 10^{-6}} = 1.5625 \times 10^5 < 5 \times 10^5 \quad \therefore \quad \text{Laminar}$$

$\therefore$ $$\delta = 6.325\ \text{mm},\ C_{fx} = 1.68 \times 10^{-3},\ C_{fL} = 3.36 \times 10^{-3}$$

The values for other distances are tabulated below.

Distance, m	**Re**	**δ, mm**	$\mathbf{C_{fx}}$	$\mathbf{C_{fL}}$
0.2	0.63×10^5	**4.000**	2.66×10^{-3}	5.32×10^{-3}
0.5	1.56×10^5	**6.325**	1.68×10^{-3}	3.36×10^{-3}
0.8	2.5×10^5	**8.000**	1.33×10^{-3}	2.66×10^{-3}

Note that as distance increases the local skin friction factor decreases and the average value is higher than the local value. Also note that the boundary layer thickness increases along the flow direction.

Example 6.2. ***Water*** *at 20° C flows over a flat plate at a free stream velocity of 0.2 m/s.* ***Determine the boundary layer*** *thickness and friction factors at lengths 0.2, 0.5 and 0.8 m from leading edge. The value of kinematic viscosity = 1.006×10^{-6} m^2/s, $\mu = 1.006 \times 10^{-3}$ kg/ms.*

The values calculated using equation 10.1.7, 9 and 10 are tabulated below:

Length, m	**Re**	**δ, mm**	$\mathbf{C_{fx}}$	$\mathbf{C_{fL}}$
0.2	0.40×10^5	**5.02**	3.33×10^{-3}	6.66×10^{-3}
0.5	0.99×10^5	**7.93**	2.11×10^{-3}	4.21×10^{-3}
0.8	1.59×10^5	**10.03**	1.67×10^{-3}	3.33×10^{-3}

Note the same trends as in Example 1. Also note that because of higher viscosity the friction values are higher.

6.1.5 Integral Method

In this case flow rate, momentum etc. in the boundary layer are determined using integration over the thickness of the boundary layer. The control volume chosen is shown in Fig. 6.1.5.

There is no flow through the face ad. (consider unit plate width)

Flow through face $$ab = \int_0^H \rho u\, dy$$

Flow through face $cd = \int_0^H \rho u dy + \frac{d}{dx}\left[\int_0^H \rho u dy\right] dx$

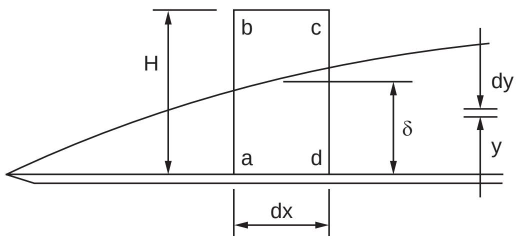


Figure 6.1.5 Boundary layer element for integral analysis

The difference should flow through *bc* as no flow is possible across *ad*.

$\therefore$ Flow through face $bc = -\frac{d}{dx}\left[\int_0^H \rho u dy\right] dx$

This is the result of continuity principle. Considering *x* directional momentum,

Momentum flow through $ab = \int_0^H u \rho u dy$

Momentum flow through $cd = \int_0^H u\rho u dy + \frac{d}{dx}\left[\int_0^H u\rho u dy\right] dx$

The mass crossing the boundary *bc* has a velocity of u_∞

Momentum flow through $bc = -\frac{d}{dx}\left[\int_0^H u_\infty \rho u dy\right] dx$

Summing up, the net momentum flow through the control volume

$$= \frac{d}{dx}\left[\int_0^H (u - u_\infty)\, \rho u dy\right] dx \qquad \text{... (1)}$$

As $(u - u_\infty)$ is zero beyond δ, the integration limit can be taken as δ instead of *H*. It is assumed that there is no pressure gradient in the boundary layer. The velocity gradient at face *bc* is zero. So the only force on the control volume surface is

$$-\tau_w\, dx = -\mu \frac{du}{dy} dx, \text{ Equating}$$

$$\frac{d}{dx}\left[\int_0^\delta (u_\infty - u)\, \rho u dy\right] = \mu \left.\frac{du}{dy}\right|_{y=0}$$

or $$\frac{d}{dx}\left[u_\infty^2 \int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy\right] = v\left.\frac{du}{dy}\right|_{y=0}$$

This is called momentum integral equation. The boundary conditions are

at $y = 0$, $u = 0$; at $y = \delta$, $u = u_\infty$ and $\dfrac{du}{dy} = 0$

Also $\dfrac{d^2u}{dy^2} = 0$ at $y = 0$ (constant pressure gradient)

Equation 10.1.12 can be solved if a velocity profile satisfying the boundary conditions is assumed. Out of the popularly used profiles the results obtained from a cubic profile given below is in closer agreement with the exact solution.

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left[\frac{y}{\delta}\right]^3$$

Substituting in equation 10.1.12

$$\frac{d}{dx}\left\{u_\infty{}^2 \int_0^\delta \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] \times \left[1 - \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy\right\} = v\left.\frac{du}{dy}\right|_{y=0}$$

Carrying out the integration, gives

$$\frac{d}{dx}\left[\frac{39}{280}u_\infty^2\,\delta\right] = \frac{3}{2}v\frac{u_\infty}{\delta}$$

or $$\frac{39}{280}u_\infty^2\frac{d\delta}{dx} = \frac{3}{2}v\frac{u_\infty}{\delta}$$, Separating variables and integrating

$$\int_0^x \delta\, d\delta = \int_0^x \frac{140}{13}\frac{v}{u_\infty}dx$$ at $x = 0$, $\delta = 0$. This leads to

$$\delta = 4.64\,x\sqrt{\frac{v}{u_\infty x}} = 4.64x/\mathrm{Re}_x^{0.5}$$

This solution is closer to the exact solution where the constant is 5 instead of 4.64. The value of C_{fx} can be determined using the assumed velocity profile.

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left[\frac{y}{\delta}\right]^3, \quad \therefore \left.\frac{du}{dy}\right|_{y=0} = u_\infty\left[\frac{3}{2\delta}\right] \quad \therefore \quad \tau_w = \mu u_\infty\left[\frac{3}{2\delta}\right]$$

$$C_f = \tau_w/\{(1/2)\,\rho\, u_\infty{}^2\} = \frac{3\mu u_\infty}{2\delta}\frac{2}{\rho u_\infty^2}$$ As $\delta = 4.64x/\mathrm{Re}^{1/2}$

$\therefore$ $$C_{fx} = \frac{3\mu u_\infty}{2 \times 4.64 \times x}\mathrm{Re}_x^{0.5}\frac{2}{\rho u_\infty^2}$$

$$C_{fx} = \frac{3}{4.64}\frac{\mu}{\rho u_\infty x}\mathrm{Re}_x^{0.5} = 0.646/\mathrm{Re}_x{}^{1/2}$$

Compared to $0.664/\mathrm{Re}_x^{1/2}$ by exact solution.

Due to flexibility this method becomes more versatile as compared to the exact method. Analysis using linear and sine function profiles illustrated under solved problems.

6.1.6 Displacement Thickness

Compared to the thickness δ in free stream, the flow in the boundary layer is reduced due to the reduction in velocity which is the result of viscous forces. In the absence of the boundary layer the flow rate that would pass through the thickness δ will be higher. The idea is illustrated in Fig. 10.1.6.

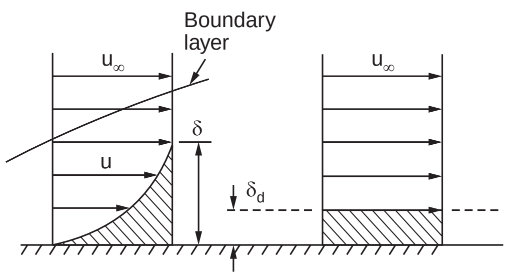


Figure 6.1.6 Displacement thickness

The reduction in volume flow is given by (for unit width)

$$= \int_0^{\delta} \rho\,(u_\infty - u)\,dy$$

If viscous forces were absent the velocity all through the thickness δ will be equal to u_∞. A thickness δ_d can be defined by equating the reduction in flow to a uniform flow with velocity u_∞ or $\rho u_\infty \delta_d$

$$\delta_d = \int_0^{\delta} \frac{(u_\infty - u)}{u_\infty}\,dy = \int_0^{\delta}\left(1 - \frac{u}{u_\infty}\right)dy$$

Displacing the boundary by a distance δ_d would pass the flow in the boundary layer at free stream velocity.

Displacement thickness δ_d is the distance by which the solid boundary would have to be displaced in a frictionless flow to give the same mass flow rate as with the boundary layer.

The displacement thickness will equal δ/3. The can be shown by assuming polynomial variation for velocity u in the boundary layer. Assuming (as there are three boundary conditions) the distribution,

$$u = a + by + cy^2, \text{ with boundary conditions,}$$

(i) $u = 0$ at $y = 0$, (ii) $u = u_\infty$ at $y = \delta$ and $(du/dy) = 0$ at $y = \delta$

The first condition gives $a = 0$ and from the other two conditions

$$c = -\,u_\infty/\delta^2 \quad \text{and} \quad b = 2u_\infty/\delta$$

Hence the profile is $\frac{u}{u_\infty} = 2\frac{y}{\delta} - \left[\frac{y}{\delta}\right]^2$

Note that this is different from the profile previously assumed for the solution of momentum integral equation. Substituting in (10.1.17) and integrating,

$$\delta_d = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy = \int_0^\delta \left\{1 - \frac{2y}{\delta} + \left(\frac{y}{\delta}\right)\right\} dy = \left[y - \frac{y^2}{\delta} - \frac{1}{3}\frac{y^3}{\delta^2}\right]_0^\delta = (1/3)\delta$$

i.e., $\delta_d = \delta/3$ or displacement thickness equals one third of hydrodynamic boundary layer thickness. In case other profiles are adopted, this constant will be different. But this is the value nearer the Blasius solution.

Example 6.3. *Using data of problems* ***Examples 6.1 and 6.2 determine the displacement thickness*** *at the various locations. Also determine the flow out of the boundary layer in the y direction and the average values of velocity v in these sections.*

The deficit flow should go out of the top of the boundary layer. From example 6.1 **air flow at** 30 °C with free stream velocity 5 m/s, (unit width is assumed)

Distance	δ, mm	δ_d, mm	Volume flow, m³/s	$V_{(0-x)}$, m/s
0.2	4.0	**1.333**	$1.333 \times 5 \times 10^{-3}$	0.0333
0.5	6.325	**2.108**	$2.108 \times 5 \times 10^{-3}$	0.0211
0.8	8.00	**2.666**	$2.666 \times 5 \times 10^{-3}$	0.0167

The volume flow out (deficit flow) equals $\delta_d\, u_\infty \times$ width, assuming 1 m width

∴ between $x = 0$ and $x = 0.2$ flow is $1.333 \times 5 \times 10^{-3}$ m³/s.

The average velocity, V = volume/area, Area = 1×0.2 m².

∴ $V = 1.333 \times 5 \times 10^{-3}/0.2 = 0.0333$ m/s. For other lengths values are tabulated above. In the case of **example 6.2, water flow** the values are given below,

Distance, m	δ, mm	δ_d, mm	flow rate, m³/s	$V_{(0-x)}$, m/s
0.2	5.02	**1.673**	3.35×10^{-4}	1.67×10^{-3}
0.5	7.93	**2.643**	5.29×10^{-4}	1.06×10^{-3}
0.8	10.03	**3.343**	6.69×10^{-4}	0.84×10^{-3}

6.1.7 Momentum Thickness

Similar to the conditions discussed in section (6.1.6) for displacement thickness, there is a reduction in momentum flow through the boundary layer as compared to the momentum flow in a thickness δ at free stream velocity.

The thickness which at free stream velocity will have the same momentum flow as the dificit flow is called momentum thickness. The deficit flow at any thin layer at y of thickness dy is (for unit width) $\rho\,(u_\infty - u)\,dy$

Momentum for this flow is $\rho u\,(u_\infty - u)\,dy$

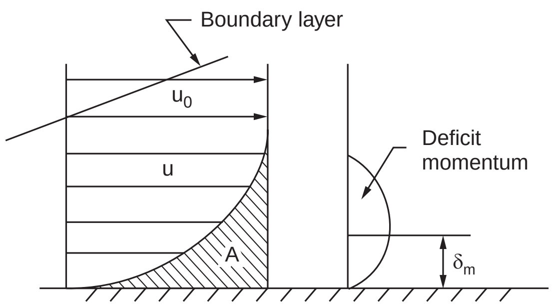
Hence the deficit momentum = $\int_0^{\delta} \rho u\,(u_\infty - u)\,dy$

Considering δ_m as momentum thickness,

$$\delta_m \ \rho u_\infty \ u_\infty = \int_0^{\delta} \rho u\,(u_\infty - u)\,dy$$

$$\delta_m = \int_0^{\delta} \left[\frac{u}{u_\infty} - \left(\frac{u}{u_\infty}\right)^2\right] dy = \int_0^{\delta} \frac{u}{u_\infty}\left[1 - \frac{u}{u_\infty}\right] dy$$

The concept of reduction in momentum is shown in Fig. 10.1.7.



The value of momentum thickness is generally taken as 1/7th of boundary layer thickness in laminar flow. The value will vary with the assumption about velocity distribution. For example if the velocity profiles as in the previous article is used, then

$$\frac{u}{u_\infty} = 2\,\frac{y}{\delta} - \left[\frac{y}{\delta}\right]^2 \quad \text{substituting in 10.1.19 and simplyifying}$$

$$\delta_m = \int_0^{\delta} \left[2\,\frac{y}{\delta} - 5\left[\frac{y}{\delta}\right]^2 + 4\left[\frac{y}{\delta}\right]^3 - \left[\frac{y}{\delta}\right]^4\right] dy$$

$$= \delta - \frac{5}{3}\delta + \delta - \frac{1}{5}\delta = \frac{2}{15}\delta = \left(\frac{1}{7.5}\right)\delta$$

6.2 TURBULENT FLOW

As flow preceeds farther along the flat plate, inertia forces begin to prevail and viscous forces are unable to keep the flow in an orderly way. **Reynolds number is the ratio of inertia force to viscous force. As inertia force increases Reynolds number increases and the flow becomes turbulent**. Generally the limiting Reynolds number for laminar flow over flat plate is taken as 5×10^5 (for internal flow the critical Reynolds number is 2000).

Turbulent flow is characterized by the variation of velocity with time at any location. The velocity at any location at any time, can be represented by

$$u = \bar{u} + u'$$

where u is the instantaneous velocity, $\bar{u}$ is the average over time and u' is the fluctuating component. The flow is steady as u' is constant at any location. An accurate velocity profile known as universal velocity profile, having different distributions at different heights is available. However it is too cmplex for use with integral method at our level of discussion.

One seventh power law has been adopted as a suitable velocity distribution for turbulent flow.

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$$

Substituting in the integral momentum equation 10.1.2, boundary layer thickness is obtained as

$$\delta = 0.382\, x/\mathrm{Re}_x^{\,0.2} \qquad (6.2.2)$$

For combined laminar and turbulent flow,

$$\delta_L = (0.381x/\mathrm{Re}_L^{\,0.2}) - (10256/\mathrm{Re}_L) \qquad (6.2.2\ a)$$

The friction coefficient is obtianed as

$$C_{fx} = 0.0594/\mathrm{Re}_x^{\,0.2} \qquad (6.2.3)$$

for combined laminar turbulent flow

$$C_{fL} = 0.074\mathrm{Re}^{-0.2} - 1742\mathrm{Re}_L^{\,-1} \qquad (6.2.4)$$

Displacement thickness is obtained as $\delta_d = \delta/8$

Example 6.4. ***Water flows*** *at a velocity of 1.2 m/s over a flat plate 1.2 m long. Assume 1/7th power law and* ***determine the boundary layer thickness and displacement thickness.*** *Compare the values with values calculated using laminar flow correlations.*

$v = 1.006 \times 10^{-6}\ m^2/s.$

$$\mathrm{Re} = \frac{ux}{v} = \frac{1.2 \times 1.2}{1.006 \times 10^{-6}} = 1.43 \times 10^6 > 5 \times 10^5 \text{ So the flow is turbulent}$$

$\boldsymbol{\delta_L} = 0.382x/Re_L^{\,0.2} = 0.0269\ m$ or **26.9 mm**

$$\delta_d = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy = \int_0^\delta \left(1 - \left(\frac{y}{\delta}\right)^{1/7}\right) dy = \left[y - \frac{7}{8}\frac{y^{1+\frac{1}{7}}}{\delta^{1/7}} \right]_0^\partial = \frac{1}{8}\delta$$

$\delta_{dL} = 26.9/8 = 3.37$ mm, $C_f = 0.0594/\mathrm{Re}^{0.2} = 0.003488$

$\tau_w = C_f (1/2)\, \rho\, u_\infty^{\,2} = (0.003488/2) \times 1000 \times 1.2^2 = 2.51\ \mathrm{N/m^2}$

In case laminar flow correlations were used:

$\delta = 5x/\mathrm{Re}^{0.5} = 0.005$ m or 5.0 mm (about 1/5th)

$\delta_d = \delta/7 = 0.72$ mm, $C_f = 0.664/\mathrm{Re}^{0.5} = 5.55 \times 10^{-4}$

$\tau = 5.55 \times 10^{-4} \times 0.5 \times 1000 \times 1.2^2 = 0.40\ \mathrm{N/m^2}$

The boundary layer is thicker and shear stress is higher in turbulent flow.

6.3 FLOW SEPARATION IN BOUNDARY LAYERS

Boundary layer is formed in the case of flow of real fluids. Viscous forces exist in such flows. The shear stress at the wall is given by

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0}, \text{ The wall shear cannot be zero. Hence at } y = 0, \ \frac{du}{dy}$$

cannot be zero. This means that the velocity gradient at the wall cannot be zero.

Separation of flow is said to occur when the direction of the flow velocity near the surface is opposed to the direction of the free stream velocity, which means $(du/dy) \leq 0$. Such a situation does not arise when there is no pressure gradient opposed to the flow direction, *ie.,* the pressure downstream of flow is higher compared to the pressure upstream. An example is subsonic diffuser. In the direction of flow the pressure increases. The increase in area along the flow causes a pressure rise.

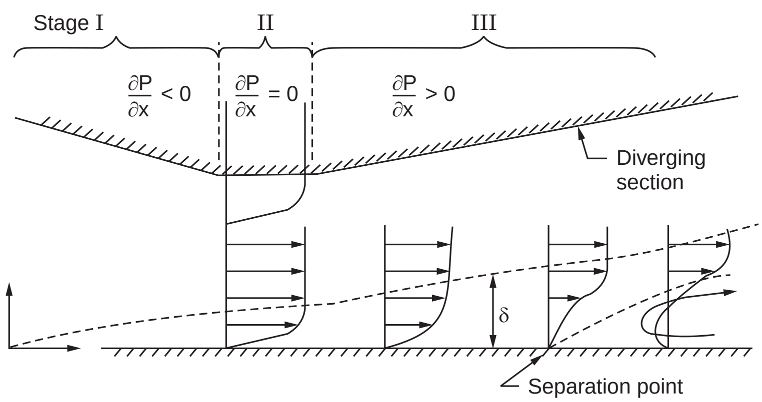


Figure 6.3.1 Flow separation

If (dp/dx) increases to the extent that it can overcome the shear near the surface, then separation will occur. Such a pressure gradient is called adverse pressure gradient. In the case of incompressible flow in a nozzle a favorable pressure gradient exists. Separation will not occur in such flows. In the case of diverging section of a diffuser, separation can occur if the rate of area increase is large. This is shown in Fig. 6.3.1. In turbulent flow, the momentum near the surface is high compared to laminar flow. Hence turbulent layer is able to resist separation better than laminar layer.

In the case of flow over spheres, cylinders, blunt bodies, airfoils etc., there is a change in flow area due to the obstruction and hence an adverse pressure gradient may be produced. Simple analytical solutions are not available to determine exactly at what conditions separation will occur. Experimental results are used to predict such conditions.

6.3.1 Flow Around Immersed Bodies – Drag and Lift

When fluid flows around a body or the body moves in a fluid there is a relative motion between the fluid and the body. The body will experience a force in such a situation. In the case of a flat plate positioned parallel to the direction of the flow, the force is parallel to the surface.

But generally in the case of blunt bodies, the force will neither be paraller nor perpendicular to the surface. The force can be resolved into two components one parallel to the flow and the other perpendicular to the flow. The former may be called shear force and the other, the pressure force.

The component parallel to the direction of motion is called drag force F_D and the component perpendicular to the direction of motion is called lift force, F_L. Determination of these forces is very important in many applications, an obvious example being aircraft wings. Simple analytical methods are found to be insufficient for the determination of such forces. So experimentally measured coefficients are used to compute drag and lift.

6.3.2 Drag Force and Coefficient of Drag

Drag is the component of force acting parallel to the direction of motion. Using the method of dimensional analysis the drage force can be related to flow Reynolds number by

$$\frac{F_D}{\rho AV^2} = f(\mathrm{Re}) \quad (6.3.1)$$

For generality velocity is indicated as V

Defining coefficient of drag as the ratio of drag to dynamic pressure, it is seen that

$$C_D = f(\mathrm{Re}),$$

$$C_D = \frac{F_D}{(1/2)\rho AV^2} \quad (6.3.2)$$

This applies to viscous drag only. In case wave drag is encountered, then

$$C_D = f\ (Re,\ Fr) \quad (6.3.3)$$

If compressibility effect is to be considered

$$C_D = f(Re, M) \quad (6.3.4)$$

Friction coefficient over flat plate in laminar flow, at a location was defined by $C_{fx} = \tau_w /(1/2)\,\rho\, A\, V^2 = 0.664/Re_x^{0.5}$. Over a given length the average value is obtained as twice this value. For a flat plate of length L, in laminar flow

$$C_D = 1.328/Re_L^{0.5} \quad (6.3.5)$$

In turbulent flow in the range $5 \times 10^5 > Re < 10^7$

$$C_D = 0.074/ Re_L^{0.2} \quad (6.3.6)$$

For Re_L up to 10^9, an empirical correlation due to Schlichting is

$$C_D = 0.455/(\log Re_L)^{2.58} \quad (6.3.7)$$

For combined laminar and turbulent flow in the range $5 \times 10^5 > Re < 10^7$

$$C_D = \frac{0.074}{\mathrm{Re}_L^{0.2}} - \frac{1740}{\mathrm{Re}_L} \quad (6.3.8)$$

For the range $5 \times 10^5 > Re < 10^9$

$$C_D = \frac{0.455}{(\log \mathrm{Re}_L)^{2.58}} - \frac{1610}{\mathrm{Re}_L} \quad (6.3.9)$$

The values of C_D for laminar flow is in the range 0.002 to 0.004.

Example 6.5. *A ship having a wetted perimeter of 50 m and length of 140 m is to travel at 5 m/s.* ***Determine the power required to overcome the skin friction****. Assume kinematic viscosity v = 1.4 × 10^{-6} m^2/s. Density 1025 kg/m^3*

$$Re = 5 \times 140/1.4 \times 10^{-6} = 0.5 \times 10^9,$$

So the equation applicable is 10.3.9

$$C_D = \frac{0.455}{(\log 0.5 \times 10^9)^{2.58}} - \frac{1610}{0.5 \times 10^9} = 1.719 \times 10^{-3}$$

$$F_D = C_D\, A(1/2)\, \rho u^2 = (1.7179 \times 10^{-3})\,(1/2) \times 140 \times 50 \times 1025 \times 5^2 \text{ N}$$

$$= \mathbf{0.154 \times 10^6\ N}$$

$$\therefore \quad \textbf{Power} = F_D\, u = 0.154 \times 10^6 \times 5 = 0.77 \times 10^6 \text{ W} = \mathbf{0.77\ MW}$$

6.3.3 Pressure Drag

When flow is perpendicular to blunt objects, like a plate or a disk, shear does not contribute to drag force. The drag is then mainly due to pressure difference between the faces. So it is called pressure drag. The drag coefficient is based on the frontal area (or projected area) of the object. In the case of airfoils the plan area is the basis for drag coefficient. The drag coefficient for same geometries are shown in Table 6.3.1 below. These are applicable for Re > 10^3.

Table 6.3.1 Drag coefficients for various shapes

Shape	$\mathbf{C_D}$
Square plate	1.18
Rectangle 1:5	1.20
Cube	1.05
Disk	1.17
Hemisphere facing flow	1.42
Parachute	1.20
Hemisphere facing downstream	0.38

It may be seen that the coefficient of pressure drag is independent of Reynolds number.

Example 6.6. *A drag chute is used to slowdown a car with a mass 1800 kg travelling at 60 m/s. The value of coefficient of drag for the car is 0.32 and frontal area is 1.1 m^2. The chute is of 1.8 m* ***diameter and drag coefficient is*** *1.2 Density of air = 1.2 kg/m^3. Determine the speed after 50 secs. Also determine the time for the speed to reach 20 m/s.*

The total drag force at any instant for the car and the chute is given by (subscript C refers to car and P refers to parachute)

$$F_D = \frac{1}{2}\rho u^2\,[C_{DC}A_C + C_{DP}A_P] \text{ and this force acts to decelerate the car.}$$

Force = mass × Acceleration = $m(du/dt)$ ∴ (du/dt) = force/mass

$$\therefore \quad \frac{du}{dt} = \frac{F_D}{m} = -\frac{k}{m}u^2 \text{ where } k = \frac{\rho}{2}\ [C_{DC}A_C + C_{DP}A_P]$$

Separating variables and integrating,

$$\int_{u_0}^{u} \frac{du}{u^2} = -\frac{k}{m}\int_0^t dt$$

$$\therefore \quad \frac{1}{u_0} - \frac{1}{u} = -\frac{k}{m}t$$

$$\therefore \quad u = \frac{u_0}{1+\left(\frac{k}{m}\right)u_0 t} \qquad \text{... }(A)$$

$$k = \frac{1.2}{2}\ [(0.32 \times 1.1) + (1.2 \times \pi \times 1.8^2/4)] = 2.0433$$

∴ $(k/m)u_0$ = (2.0433 × 60)/1800 = 0.06811

∴ (*i*) After 50 seconds, **u** = 60/(1 + 0.06811 × 50) = **13.62 m/s**

(*ii*) For u = 20, 20 = 60/(1 + 0.06811 × t) ∴ **t = 29.36 seconds**

The distance travelled can be obtained by integrating $u\ dt$.

$$\therefore \quad s = \int_0^t u\,dt = \int_0^t \frac{u_0 dt}{1+(k/m)u_0 t} = \frac{u_0}{(k/m)u_0}\ \ln\ [1 + (k/\mathrm{m})u_0 t] = (m/k)\ \ln\ [1 + (\mathrm{k/m})u_0 t]$$

At t = 50 sec $s = \dfrac{1800}{2.0433}$ ln (1 + 0.06811 × 50) = 1306 m

At t = 29.36 sec $s = \dfrac{1800}{2.0433}$ In (1 + 0.06811 ×29.36) = 968 m

6.3.4 Flow Over Spheres and Cylinders

In these cases both pressure and friction drag contribute to the total drag. The flow separation at the rear and formation of wake contributes to the pressure drag. The flow pattern and the variation of drag coefficient is shown in Fig. 6.3.2. It may be noted that the coefficient of drag is nearly constant from $Re = 10^3$ to 5×10^5. From experiments the boundary layer in the forward portion is found to be laminar in this range. Separation is found to occur at about mid section and a wide wake is found to exist with pressure in the wake below that at the front.

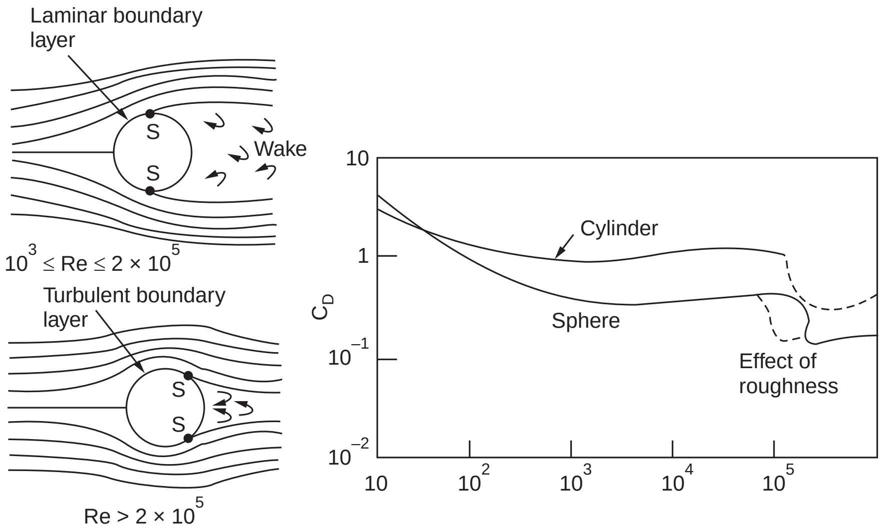


Figure 6.3.2 Flow separation in flow over cylinder/sphere

There is a sharp drop in the value of C_D after the critical Reynolds number. The flow in the forward side is found to turn turbulent and separation moves downstream and wake is now narrow, reducing the net pressure drag leading to the abrupt decrease in the drag coefficient. Turbulent layer has a higher momentum near the surface resisting separation.

Separation can be reduced by streamlining the body shape, reducing the pressure drag. This generally increases the area thus increasing friction drag. An optimum streamlined shape is the one which gives minimum total drag. Stream lining is now adopted not only for aircrafts but almost for all transport vehicles.

Example 6.7. *A model of a bathysphere 50 mm diameter is towed under water at a speed of 1 m/s.* ***Determine the tension in the towline****. Density of water = 1020 kg/m ^{3}. Kinematic viscosity = 1.006×10^{-6} m^2/s*

$$Re = uD/v = 1 \times 0.05/1.006 \times 10^{-6} = 4.97 \times 10^4$$

From graph (Fig. 10.3.2) C_D is read as 0.45

$$\therefore \quad \mathbf{F_D} = C_D\,(1/2)\,\rho\,Au^2 = \frac{0.45}{2} \times 1020 \times \frac{\pi \times 0.05^2}{4} \times 1^2 = \mathbf{0.45\ N}$$

6.3.5 Lift and Coefficient of Lift

The force on an immersed body moving in a fluid can be resolved into two components. The component along the flow direction is called drag. The component perpendicular to the flow direction is called lift. The lift on airfoil is an example. The coefficient of lift is defined by

$$C_L = \frac{F_L}{(1/2)\,\rho\,Au^2} \tag{6.3.10}$$

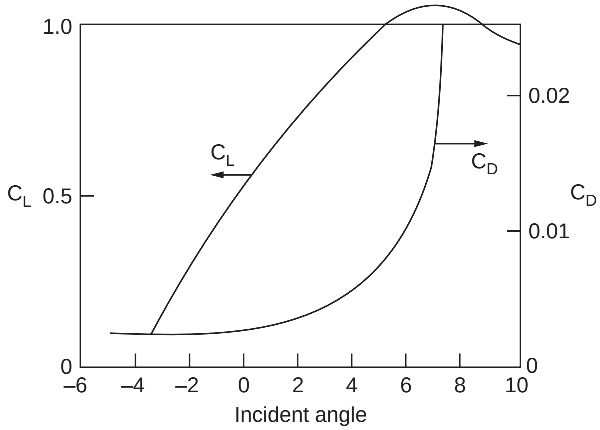


Figure 6.3.3 Variation of Lift and Drag on an airfoil

Lift is of interest mainly in the design of airfoil sections. Airfoil blade shapes are also used in turbomachines. The lift and drag coefficients depend on the Reynolds number and angle of attack. The angle between the airfoil chord and the flow direction is called angle of attack. The chord of an airfoil is the line joining the leading edge and the trailing edge. The planform area (the maximum projected area) is used in the definition of lift and drag coefficients. A typical plot of the variation of lift and drag coefficients with angle of attack for a specified Reynolds number is shown in Fig. 6.3.3.

For each airfoil section such plots are available. Flow separation will result in sudden drop in the lift, known as stall. Presently computer softwares are available for the design of airfoil sections with a very high ratio of lift to drag. These data are for long spans and corrections should be made as per the aspect ratio defined by b^2/A_p. where b is the span length and A_p is the planform area. This will equal the ratio (span/chord) as, $A_p = bc$. The lift to drag ratio varies from 20 to 40 with the lower value applicable for small planes.

6.3.6 Rotating Sphere and Cylinder

In order to reduce skin friction in flow over surfaces, particularly curved surfaces boundary layer control is used. One method of boundary layer control is by the use of moving surfaces at locations where separation may start. This is difficult to apply due to mechanical restrictions. However this principle is used in sports like baseball, golf, cricket and tennis where spin is applied to control the trajectory of the ball. Spin also provides significant aerodynamic lift to increase the distance travelled by the ball. Spin can also be used to obtain a curved path of travel for the ball.

Spin alters the pressure distribution and also the location of boundary layer separation. For spin along the flow direction, separation is delayed on the upper surface and it occurs earlier in the lower surface. Pressure is reduced on the upper surface and is increased on the lower surface and the wake is deflected downwards.

The coefficients of lift and drag are found to be a function of ω D/2u called spin ratio.

Example 6.8. ***Show*** *using dimensional analysis that the* ***lift and drag coefficients are functions of spin ratio and Reynolds number****.*

The variables affecting the phenomenon are listed below. As C_L and C_D are dimensionless, these are not listed.

No	Variable	Unit	Dimension
1	Linear Velocity, u	m/s	L/T
2	Radius, R	m	L
3	Angular velocity, ω	Radians/s	$1/T$
4	Kinematic viscosity, v	m^2/s	L^2/T

There are four variables and two dimensions, namely L and T. Hence two π terms can be identified. Choosing linear velocity and radius as repeating variables

Let $$\pi_1 = \omega\, u^a R^b \text{ or } L^0 T^0 = \frac{1}{T}\,\frac{L^a}{T^a}\,L^b$$

$\therefore$ $$a + b = 0, -1 - a = 0 \quad \therefore \quad a = -1 \quad \therefore \quad b = 1$$

$$\pi_1 = \omega\, R/u = \omega\, R/2u, \text{ called } \textbf{spin ratio.}$$

Let $$\pi_2 = vu^a R^b \quad \text{or} \quad L^0 T^0 = \frac{L^2}{T}\,\frac{L^a}{T^a}\,L^b, \quad \therefore \quad 2 + a + b = 0, -1 - a = 0$$

$\therefore$ $$a = -1 \quad b = -1 \quad \pi_2 = \frac{v}{uR} \quad \text{or} \quad \frac{uD}{v}, \text{ Reynolds number.}$$

Hence $$\mathbf{C_L = f\left[\frac{\omega\, D}{2u}, \frac{uD}{v}\right]} \text{ and } \mathbf{C_D = f\left[\frac{\omega\, D}{2u}, \frac{uD}{v}\right]}$$

The variation of C_L and C_D are found to be influenced more by spin ratio than Reynolds number. The trend is shown in Fig. Ex. 6.8. In the case of cylinders the area for definition of C_L and C_D is $L \times D$

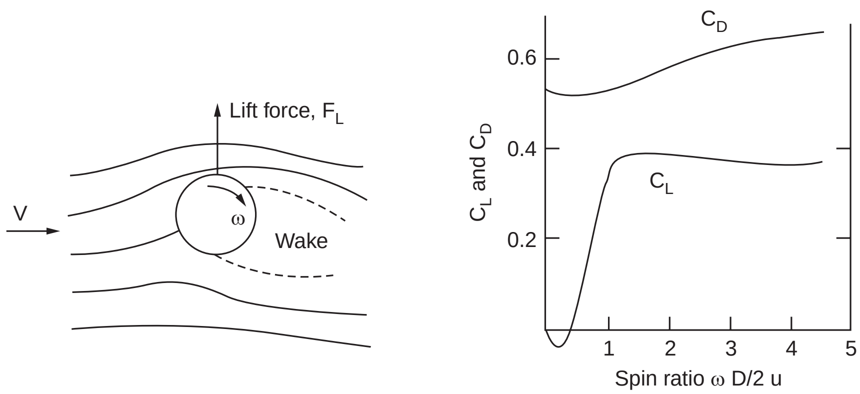


Figure Ex. 6.8 Variation of Lift and Drag with spin ratio

A force perpendicular to both direction of motion and the spin axis is created during the flight. This is known as MAGNUS effect. This can cause drift in the flight path.

SOLVED PROBLEMS

Problem 6.1 *Assuming* ***linear velocity variation in the boundary layer*** *and using linear momentum integral equation, determine the thickness of the boundary layer. Also determine the friction coefficient and the displacement and momentum thicknesses.*

Momentum integral equation is

$$\frac{d}{dx}\left[\int_0^\delta u(u_\infty - u)dy\right] = v\left.\frac{du}{dy}\right|_{y=0}. \text{ As } \frac{u}{u_\infty} = \frac{y}{\delta} \quad \therefore \quad u = \frac{u_\infty y}{\delta},$$

$$\frac{du}{dy} = \frac{u_\infty}{\delta} \quad \therefore \ \tau = \frac{\mu u_\infty}{\delta}, \text{ Considering the integral part}$$

$$\int_0^\delta \left[\frac{u_\infty^2}{\delta} y - \frac{u_\infty^2}{\delta^2} y^2\right] dy = \left[\frac{u_\infty^2 \delta}{2} - \frac{u_\infty^2 \delta}{3}\right] = \frac{1}{6} u_\infty^2 \delta$$

$$\therefore \qquad \frac{d}{dx}\left[\frac{u_\infty^2}{6}\delta\right] = v\frac{u_\infty}{\delta} \text{ or } \frac{u_\infty^2}{6}\frac{d\delta}{dx} = v\frac{u_\infty}{\delta}$$

Separating variables and integrating, $\delta\, d\delta = (6v/u_\infty)\, dx$

$$\delta^2 = (12vx)/u_\infty = 12x^2/(v/u_\infty x) = 12x^2/\text{Re}_x$$

$$\therefore \qquad \boldsymbol{\delta = 3.464x/\text{Re}_x^{0.5}},$$

The constant is 3.464 instead of 5 in the exact solution

$$\mathbf{C_{fx}} = \frac{\tau}{(1/2)\rho u_\infty^2} = \frac{2\mu u_\infty}{\rho u_\infty^2 \delta} = \frac{2v\,\text{Re}_x^{0.5}}{u_\infty 3.464x} = \mathbf{0.577/Re_x^{0.5}}$$

The displacement thickness

$$\delta_d = \int_0^\delta \left[1 - \frac{u}{u_\infty}\right] dy = \int_0^\delta \left[1 - \frac{y}{\delta}\right] dy = (1/2)\delta \text{ or } \delta/2. \text{ As against } \delta/3$$

Momentum thickness is given by

$$\delta_m = \int_0^\delta \frac{u}{u_\infty}\left[1 - \frac{u}{u_\infty}\right] dy = \int_0^\delta \left[\frac{y}{\delta} - \frac{y^2}{\delta^2}\right] dy = \frac{1}{2}\delta - \frac{1}{3}\delta = \frac{1}{6}\delta$$

By the exact solution, $\delta_m = (1/7)\delta$

Problem 6.2 *Assuming* ***second degree velocity distribution in*** *the boundary layer determine using the integral momentum equation, the thickness of boundary layer friction coefficient, displacement and momentum thicknesses.*

Let $u = a + by + cy^2$. The boundary conditions are $u = 0$ at $y = 0$,

At $y = \delta$, $\dfrac{du}{dy} = 0$, and $u = u_\infty$. This gives $\dfrac{u}{u_\infty} = 2\dfrac{y}{\delta} - \left(\dfrac{y}{\delta}\right)^2$

Substituting in the integral momentum equation,

$$\frac{d}{dx}\left[\int_0^{\delta} u(u_\infty - u)dy\right] = v\left.\frac{du}{dy}\right|_{y=0},$$

$$u = u_\infty\left[2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right],\ \left.\frac{du}{dy}\right|_{y=0} = 2u_\infty/\delta,\ \tau = 2\mu u_\infty/\delta$$

Considering the integral part,

$$u_\infty^2\int_0^{\delta}\left[\left(\frac{u}{u_\infty}\right) - \left(\frac{u}{u_\infty}\right)^2\right]dy = u_\infty^2\int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 - 4\left(\frac{y}{\delta}\right)^2 + 4\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right]dy$$

$$= u_\infty^2\int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - 5\left(\frac{y}{\delta}\right)^2 + 4\left(\frac{y}{\delta}\right)^3 - \left(\frac{y}{\delta}\right)^4\right]dy$$

$$= u_\infty^2\left[\delta - \frac{5}{3}\delta + \delta - \frac{1}{5}\delta\right] = \frac{2}{15}u_\infty^2\delta \qquad \ldots (A)$$

Substituting

$$\frac{d}{dx}\left[\frac{2}{15}u_\infty^2\delta\right] = 2vu_\infty/\delta \quad \text{or} \quad \frac{2}{15}u_\infty^2 = \frac{d\delta}{dx} = 2\,vu_\infty/\delta$$

$\therefore$ $\delta d\delta = 15(v/u_\infty)dx$ Integrating

$$\delta^2 = 30vx/u_\infty,\ 30\left(\frac{v}{u_\infty x}\right)x^2 = 30x^2/\mathrm{Re}_x,\ \boldsymbol{\delta = 5.477x/\mathrm{Re}_x^{0.5}}$$

Note that the constant is 5.477 as against 5 by exact solution.

As $\tau = 2\mu u_\infty/\delta$ and $\delta = 5.477\ x/\mathrm{Re}_x^{0.5}$

$$\mathbf{C_{fx}} = \frac{\tau}{(1/2)\rho u_\infty^2} = \frac{4\mu u_\infty}{\rho u_\infty^2\delta} = \frac{4v}{u_\infty\delta} = \frac{4v\,\mathrm{Re}_x^{0.5}}{5.477\,xu_\infty} = \mathbf{0.73/Re_x^{0.5}}$$

instead of $0.644/\mathrm{Re}_x^{0.5}$

$$\delta_d = \int_0^{\delta}\left[1 - \frac{u}{u_\infty}\right]dy = \int_0^{\delta}\left[1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right]dy = \frac{\delta}{3}$$

$$\delta_m = \int_0^{\delta}\left[\frac{u}{u_\infty} - \left(\frac{u}{u_\infty}\right)^2\right]dy = \frac{2}{15}\delta \quad \text{(see equation } A)$$

Problem 6.3 *Assuming the velocity distribution in the boundary layer as* $\frac{u}{u_\infty} = sin\left(\frac{\pi y}{2\delta}\right)$ *(in the range* $0 \le y \le \delta$*, and* $u/u_\infty = 1$ *beyond* δ*) determine the thickness of the boundary layer, using integral momentum method (Refer equation 10.1.12).*

$$\frac{u}{u_\infty} = \sin\left(\frac{\pi y}{2\delta}\right),\ \frac{du}{dy} = \frac{\pi}{2\delta}\cos\frac{\pi y}{2\delta}\ \text{at } y = 0,\ \frac{du}{dy} = u_\infty\ (\pi/2\delta)$$

$$\frac{d}{dx}\left[\int_0^\delta u(u_\infty - u)dy\right] = v\frac{du}{dy}\bigg|_{y=0}$$

Considering the integral part and substituting the velocity distribution,

$$u_\delta^2 \int_0^\delta \left[\left(\frac{u}{u_\infty}\right) - \left(\frac{u}{u_\infty}\right)^2\right] dy = u_\delta^2 \int_0^\delta \left[\sin\frac{\pi y}{2\delta} - \sin^2\frac{\pi y}{2\delta}\right] dy$$

Noting $\int \sin^2 ax = \frac{x}{2} - \frac{\sin 2ax}{4a}$

$$= u_\infty^2 \left[-\frac{2\delta}{\pi}\cos\frac{\pi y}{2\delta} - \frac{y}{2} + \frac{\delta}{2\pi}\sin\frac{\pi y}{2\pi}\right]_0^\delta$$

$$= u_\infty^2 \left[0 - \frac{\delta}{2} + 0\right] - \left[-\frac{2\delta}{\pi} - 0 + 0\right] = 0.1366 \times u_\infty{}^2 \times \delta \qquad \ldots (A)$$

$$\frac{d}{dx}[u_\infty^2 \times 0.1366\,\delta] = \frac{\pi}{2\delta}\,u_\infty v,\ [u_\infty^2 \times 0.1366]\,\frac{d\delta}{dx} = \frac{\pi}{2\delta}\,u_\infty v$$

$$\therefore \qquad \delta\, d\delta = \frac{\pi}{2 \times 0.1366}\,\frac{v}{u_\infty}\,dx$$

Integrating $\frac{\delta^2}{2} = \frac{\pi}{2 \times 0.1366}\,\frac{vx}{u_\infty}$ or $\boldsymbol{\delta = 4.8x/\mathrm{Re}_x{}^{0.5}}$

$\mathbf{C_{fx}} = \tau_w/(1/2)\rho\, u_\infty{}^2$, $\tau_w = \mu\,(du/dy)$, at $y = 0$, $\tau_w = \mu\, u_\infty\, \pi/2\delta$

$$\therefore \qquad \mathbf{C_{fx}} = \frac{2\mu\, u_\infty\, \pi}{2\delta\rho u_\infty^2} = \frac{\pi}{\delta}\,\frac{v}{u_\infty} = \frac{\pi}{4.8}\,\frac{v}{u_\infty x}\,\mathrm{Re}_x^{0.5} = \mathbf{0.655/Re_x{}^{0.5}}$$

$$\delta_d = \int_0^\delta \left[1 - \frac{u}{u_\infty}\right] dy = \int_0^\delta \left[1 - \sin\left(\frac{\pi y}{2\delta}\right)\right] dy = \left[\delta + \frac{2\delta}{\pi}\cos\frac{\pi y}{\delta}\right]_0^\delta$$

$= [\delta + 0] - [0 + (2\delta/\pi)] = 0.3625\,\delta = \delta/2.76$, instead of $\delta/3$

$\delta_m = 0.1366\,\delta$, or $\delta/7.32$ (refer result A)

Problem 6.4 *Using the* ***cubic velocity profile*** *determine upto a length* ***L the flow out of the boundary*** *layer in terms of the boundary layer thicknes.*

The free stream flow for thickness of δ is $\rho\, u_\infty\, \delta$.

Assuming cubic velocity profile,

$$u = u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]$$

Mass flow through the boundary layer

$$= \int_0^\delta \rho u dy = \int_0^\delta \rho u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy = \rho u_\infty \left[\frac{3}{2}\frac{y^2}{\delta} - \frac{1}{8}\frac{y^4}{\delta^3}\right]_0^\delta = \frac{5}{8}\rho u_\infty \delta$$

$\therefore$ Mass flow out of the boundary layer = (1–(5/8)) $\rho\, u_\infty\, \delta$ = 3/8 $\rho\, u_\infty\, \delta$ or displacement thickness times the free stream flow. (Nota : δ_d = (3/8) δ for cubic profile)

The average velocity in the y direction can be obtained by dividing the volume flow by area *i.e.,* 1 × x for unit width. **Volume flow out of the boundary**

$$\mathbf{v} = (3/8)\, u_\infty\, \delta, \text{ velocity} = \frac{3}{8} u_\infty \frac{4.64x}{\mathrm{Re}_L^{0.5}} \frac{1}{x} = \frac{1.74 u_\infty}{\mathrm{Re}_L^{0.5}}$$

This will be low as Reynolds number will be high.

Consider the data from example (10.1) Air flow, u_∞ = 5 m/s, at a distance 0.5 m,

$$\mathrm{Re} = 1.56 \times 10^5 \quad \therefore\ \mathbf{v} = (1.74 \times 5)/(1.56 \times 10^5)^{0.5} = \mathbf{0.022\ m/s}$$

This can also be calculated in a round about way using the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. The value of $\frac{\partial u}{\partial x}$ can be obtained from the assumed profile and then equated to $-\frac{\partial v}{\partial y}$. Integrating the same between 0 and δ the same result will be obtained. [Refer Problem 10.6].

Problem 6.5 *Using the continuity and momentum equations show that at y = 0,* $-\frac{\partial^3 u}{\partial y^3} = 0$*. Deduce from the above that the cubic profile is approximate.*

Consider the x directional momentum equation, $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2}$. Differentiating with resect to y,

$$\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = v\frac{\partial^3 u}{\partial y^3}, \text{ Simplifying.}$$

$$\frac{\partial u}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} = v\frac{\partial^3 u}{\partial y^3}$$

The first term is zero due to continuity equation. At y = 0, u = 0 and v = 0. Hence the second and third terms are also zero. So $\frac{\partial^3 u}{\partial y^3}$ should be zero.

Consider the cubic profile:

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\therefore \qquad \frac{\partial u}{\partial y} = u_\infty\left[\frac{3}{2}\frac{1}{\delta} - \frac{3}{2}\frac{y^2}{\delta^3}\right] \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = u_\infty\left[-\frac{6}{2}\frac{y}{\delta^3}\right] \frac{\partial^3 u}{\partial y^2} = -\,3u_\infty/\delta^3$$

This is not zero. Hence profile assume is approximate.

Problem 6.6 *Derive a general expression* ***for the y directional velocity at a location*** *x in the boundary layer in flow over a flat plate. Indicate at what y location this will be maximum. Assume cubic velocity variation.*

Consider continuity equation.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$$

$$u = u_\infty\left[\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right],\ \delta = 5x/\mathrm{Re}_x^{1/2} = \frac{5xv^{1/2}}{u_\infty^{1/2}\,x^{1/2}} = cx^{1/2}$$

where $c = [5v^{1/2}/u_\infty^{1/2}]$ substituting and putting $c_1 = 3u_\infty/2\,\delta$ and $c_2 = u_\infty/2\,\delta^3$, velocity expression reduces to $u = c_1\, y\, x^{-1/2} - c_2\, y^3\, x^{-3/2}$, $\dfrac{\partial v}{\partial y}=-\dfrac{\partial u}{\partial x} = -\,[c_1 \times (-1/2) \times y\, x^{-3/2} + (3/2)\, c_2\, y^3\, x^{-5/2}]$

Integrating w.r.t. y, and substituting for c_1 and c_2

$$v = \frac{3}{8c}\frac{u_\infty}{x^{3/2}}y^2 - \frac{3u_\infty}{16c^3}\frac{y^4}{x^{5/2}}\text{, Substituting for } c$$

$$v = \frac{3}{8}\frac{u_\infty u_\infty{}^{1/2}y^2}{5v^{1/2}x^{3/2}} - \frac{3\times u_\infty^{3/2}}{16\times 125v^{3/2}}\frac{u_\infty y^4}{x^{5/2}}$$

Substituting $\dfrac{5xv^{1/2}}{u_\infty^{1/2}\,x^{1/2}} = \delta$

$$v = \frac{3}{8}u_\infty\frac{y^2}{\delta x} - \frac{3}{16}u_\infty\frac{y^4}{\delta^3 x} = \frac{3}{8}\frac{u_\infty}{\delta x}\left[y^2 - \frac{1}{2}\frac{y^4}{\delta^2}\right]$$

(Check for dimensional consistency : dimensions of $y^2/\delta\, x$ and $y^4/\delta^3\, x$ cancel and v has the same unit as u_∞)

Maximum value occurs when $\dfrac{\partial v}{\partial y}=0$.

$$\frac{\partial}{\partial y}\left[y^2 - \frac{1}{2\delta^2}y^4\right] = 2y - \frac{1}{2\delta^2}4y^3\text{. Equating to zero and solving } y = \delta$$

This is physically explainable as the total flow in y direction should occur at

$$y = \delta\text{. Velocity at } y = \delta \text{ is } \mathbf{v}_{\boldsymbol{\delta}\mathbf{x}} = \frac{3}{16}\frac{u_\infty\,\delta}{x} = \mathbf{0.87}\,\frac{\mathbf{u}_\infty}{\mathbf{Re}_\mathbf{x}^{\mathbf{0.5}}}$$

Total mass flow when integrated over the length will equal (3/8) $\rho\, u_\infty\, \delta_L$ (Refer Problem 10.4).

Problem 6.7 *The shear at a location 2 m from the leading edge of a flat plate was measured as 2.1 N/m². Assuming the flow to be turbulent from the start determine if air at 20°C was flowing over the plate (i) the velocity of air (ii) the boundary layer thickness and (iii) the velocity at 15 mm above the plate.* $\rho = 1.205$ *kg/m³,* $v = 15.06 \times 10^{-6}$ *m²/s*

Using equation (10.2.3) $C_{fx} = 0.0594/\mathrm{Re}_x{}^{0.2}$, $\tau_w = C_{fx}\,(1/2)\,\rho\, u^2$,

$$\text{Equating } 2.1 = \frac{0.0594 \times (15.06 \times 10^{-6})^{0.2}}{u_\infty^{0.2}\, 2^{0.2}} \times \frac{1}{2} \times 1.205\, u_\infty^{\;2}, \text{ Solving,}$$

$$u_\infty^{\;1.8} = 621.04 \text{ or } u_\infty = 35.623 \text{ m/s}$$

$$\text{Re} = 35.623 \times 2/15.06 \times 10^{-6} = 4.808 \times 10^6,$$

Turbulent hence the use of equation (10.2.3) is justified. Using equation (10.2.1),

$$\delta = 0.382x/\text{Re}_x^{\;0.2} = 0.382 \times 2/(4.808 \times 10^6)^{0.2}$$

$$= 0.0352 \text{ m or } \delta = 35.2 \text{ mm}$$

If the velocity profile is assumed as

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$$

$\therefore$ $\quad\mathbf{u} = 35.623\ (15/35.2)^{1/7} = \mathbf{32.05\ m/s}$

Problem 6.8 ***Determine the length at which the flow over a flat plate will turn turbulent for air****, water and engine oil if the flow velocity is 3 m/s. Also determine the boundary layer thickness at the location. Temperature of the fluid = 20°C. The kinematic viscosity and density of the fluids are :*

S. No	Density, kg/m^3	Kinematic viscosity	L_{cv}, m	δ, mm
Air	1.205	15.06×10^{-6}	**2.51**	**17.7**
Water	1000	1.006×10^{-6}	**0.17**	**1.2**
Engine oil	888	901×10^{-6}	**150**	**1061**

The flow turns turbulent at Re = 5×10^5

(1) Air : $\quad 5 \times 10^5 = 3 \times L_a/15.06 \times 10^{-6} \quad \therefore \quad L_a = 2.51$ m

$$\delta = 5x/\text{Re}_x^{\;0.5} = 5 \times 2.51/(5 \times 10^5)^{0.5} = 0.0177 \text{ m}$$

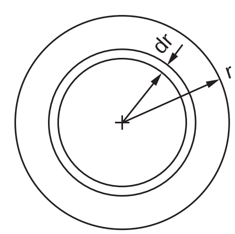
(2) Water : $\quad 5 \times 10^5 = 3 \times L_w/1.006 \times 10^{-6} \quad \therefore \quad L_w = 0.1677$ m

$$\delta = 5x/\text{Re}_x^{\;0.5} = 5 \times 0.1677/(5 \times 10^5)^{0.5} = 0.0012 \text{ m}$$

(3) Engine oil : $\quad 5 \times 10^5 = 3 \times L_o/901 \times 10^{-6} \quad \therefore \quad L_o = 150.17$ m

$$\delta = 5x/\text{Re}_x^{\;0.5} = 5 \times 150.17/(5 \times 10^5)^{0.5} = 1.061 \text{ m}$$

Problem 6.9 *The pressure distribution on the front and back surfaces of a thin disk of radius, R oriented perpendicular to a fluid stream was measured and the pressure coefficient has been correlated as below.*



Front side : $C_P = 1 - (r/R)^6$. Rear surface : $C_P = -0.42$

Determine the drag coefficient for the disk.

$$C_P = \Delta P/(1/2)\,\rho\, AV^2$$

$\therefore$ $\quad \Delta P = C_P(1/2)\,\rho\, AV^2 = C_P\, 2\pi\, rdr\, (\rho\, V^2/2)$

Consider a small strip of width dr at a radius r. The force on the area = $\Delta P \times 2\pi\, rdr$

$$F_D = \int_0^R \Delta P \times 2\pi rdr = \frac{1}{2}\rho V^2 \int_0^R C_p\ 2\pi\ rdr = \frac{1}{2}\rho V^2 \int_0^R \left[1 - \left(\frac{r}{R}\right)^6\right] 2\pi rdr$$

$$= \frac{1}{2}2\pi \times \rho V^2 \left[\frac{r^2}{2} - \frac{r^8}{8R^6}\right]_0^R = \frac{1}{2}2\pi \times \rho V^2 \left[\frac{R^2}{2} - \frac{R^2}{8}\right]$$

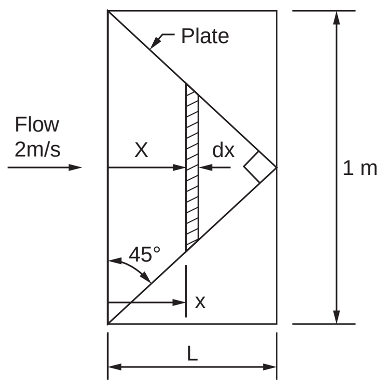
$$= \frac{1}{2}\rho V^2 \pi R^2 \, [3/4] = (1/8)\ \rho\ AV^2$$

$$\mathbf{C_D} = F_D/(1/2)\ \rho\ AV^2 \quad \therefore \mathbf{C_D} = 3/4 = \mathbf{0.75}$$

On the otherside, the pressure is independent of radius $\therefore C_D = C_p$ and it is in the opposite direction

$$\therefore \qquad C_p = 0.75 + 0.42 = \mathbf{1.17}$$

Problem 6.10 *Air flows along a triagular plate as shown in Fig. P. 10.10. Determine the shear force on both sides of the plate. Assume air temperature. as 20°C. ρ = 1.205 kg/m³, kinematic viscosity is 15.06 × 10⁻⁶ m²/s.*



Considering the maximum length of 0.5 m

$$Re = 2 \times 0.5/15.06 \times 10^{-6} = 0.66 \times 10^5 \quad \therefore \quad \text{flow is laminar}$$

$$\tau_x = 0.332\ \rho\ u^2/\text{Re}^{0.5} = 0.332\ \rho\ u^2 v^{1/2}/u^{1/2}x^{1/2}$$

$$= 0.332\ \rho\ u^{1.5} v^{1/2} x^{-1/2}$$

$$= 0.332 \times 1.205 \times 2^{1.5} \times (15.06 \times 10^{-6})^{0.5} x^{-1/2}$$

$$= 4.97 \times 10^{-3} x^{1/2}$$

Considering a strip of width dx at a distance x from base, and assuming the length of base as $2L$, height will be L.

$$dA = \frac{L - x}{L} \times 2L \times dx = 2(L - x)dx, \text{ Force on the strip}$$

$$dF = \tau_x\, dA = 4.97 \times 10^{-3} \times 2(L - x)x^{-1/2}dx$$

Integrating between $x = 0$ to $x = L$

$$F = 9.94 \times 10^{-3} \left[\frac{Lx^{1/2}}{0.5} - \frac{x^{1.5}}{1.5}\right]_0^L = 9.94 \times 10^{-3} \times \frac{4}{3} \times L^{1.5}$$

Here **L = 0.5 m.** $\therefore$ **F = 4.68 × 10⁻³ N**

Check for dimensional homogeneity.

$$F = \text{const } \rho\, u^{1.5} v^{1/2} L^{1.5},$$

$$N = \text{const } \frac{kg}{m^3} \frac{m^{1.5}}{s^{1.5}} \frac{m}{s^{0.5}}\ m^{1.5} = \text{kgm/s}^2 = \text{N},$$

Hence checks.

Problem 6.11 *A water ski is 1.2 m long and 0.2 m wide and moves in water at 10 m/s. the water temperature is 20°C.* ***Determine the viscous drag*** *approximating it as a flat plate.*

$$v = 1.006 \times 10^{-6}\ m^2/s.\ \rho = 1000\ kg/m^3,$$

$$Re = 1.2 \times 10/1.006 \times 10^{-6} = 11.93 \times 10^6$$

∴ The flow is turbulent considering combined laminar and turbulent flows.

$$C_{fL} = 0.074\text{Re}_L^{-0.2} - 1742\text{Re}_L^{-1} = 2.99 \times 10^{-3},\ \text{Drag} = C_{fL}\ (1/2)\rho\, u^2\Delta$$

$$\textbf{Drag} = (1/2)\ 1000 \times 10^2 \times 1.2 \times 0.2 \times 2.99 \times 10^{-3} = \textbf{35.88 N}$$

Power required considering 2 skis, $P = 2 \times 35.88 \times 10 = 717.6$ W

Problem 6.12 *In a power plant located near the sea* ***a chimney*** *of 1.2 m diameter and 35 m height has been installed. During a cyclone the wind reaches velocity in the range of 60 kmph.* ***Determine the moment at the base of the chimney.***

$$\rho = 1.2\ \text{kg/m}^3,\ v = 17.6 \times 10^{-6}\ \text{m}^2\text{/s},\ u = 600000/3600 = 16.67\ \text{m/s}$$

$$Re = 16.67 \times 1.2/17.6 \times 10^{-6} = 1.14 \times 10^6$$

From graph for circular cylinder C_D is read as 0.35

∴ $$F_D = C_D\, \rho\, Au^2/2 = 0.35 \times 1.23 \times 35 \times 1.2 \times 16.66^2/2 = 5022.5\ \text{N}$$

As this is a uniform force, it can be taken to act at the mid point.

∴ **Moment** = 5022.5 × 35/2 = 87893 Nm or **87.893 kNm.**

Problem 6.13 *A overhead water tank is in the shape of a sphere of 12 m diameter and is supported by a 30 m tall tower of circular section of diameter 2 m.* ***Determine the moment at the base*** *caused by the aerodynamic force due to cyclonic wind of speed 100 kmph. Assume density of air as 1.205 kg/m³ and kinematic viscosity as 15.06 × 10⁻⁶ m²/s.*

For the spherical portion : $$Re = 12 \times \frac{100 \times 1000}{3600} \times \frac{1}{1506 \times 10^{-6}} = 2.21 \times 10^7$$

The value of C_D is read as 0.19 from graph by extrapolation.

For the cylindrical portion : $$Re = 2 \times \frac{100 \times 1000}{3600} \times \frac{1}{1506 \times 10^{-6}} = 3.689 \times 10^6$$

The value of C_D is read as 0.40 from graph by extrapolation.

$$F_D = C_D (1/2)\, \rho\, AV^2,\ M = F_D \times \text{distance},$$

$$V = 100 \times 1000/3600 = 27.78\ \text{m/s}.$$

For the spherical portion

$$M = (30 + 6) \times 0.19 \times (1/2) \times 1.205 \times (\pi \times 12^2/4) \times 27.78^2$$

$$= 359.6 \times 10^3\ \text{Nm}$$

For the cylindrical portion

$$M = 15 \times 0.4 \times (1/2) \times 1.205 \times 2 \times 30 \times 27.78^2 = 167.4 \times 10^3 \text{ Nm}$$

Total moment $= (359.6 + 167.4) \times 10^3 =$ $\mathbf{527.0 \times 10^3}$ **Nm**

Problem 6.14 *A parachute moves down at a speed of 6 m/s. The mass of the chute and the jumper is 120 kg.* ***Determine the minimum diameter of the chute****. Density of air = 1.23 kg/m*^{3}.

For parachute $C_D = 1.2$, (Refer table 10.3.1)

Net force = 120×9.81 N

$120 \times 9.81 = 1.2 \times (1/2) \times 1.23 \times (\pi D^2/4) \times 6^2$ Solving **D = 7.51 m**

Problem 6.15 *Hail stones that are formed in thunder clouds are supported by the drag due to the air draft upwards and will begin to fall when the size reaches a critical value.* ***Estimate the velocity upwards so that hailstones begin to fall*** *when the diameter reaches a value of 40 mm. The density and dynamic viscosity of air at the altitude of 5000 m where the stones are formed are 0.7364 kg/m*^{3} *and* 1.628×10^{-5} *kg/ms. Hailstone is assumed to be in the shape of a sphere with a density of 940 kg/m*^{3}.

The drag force should be just less than the gravity force when the hailstone begins to fall. At the limiting condition thses can be taken as equal. Other body forces like buoyancy forces are negligible.

Drag force = C_D (1/2) $\rho A u^2$, Gravity force = ρVg, V being the volume.

Equating and substituting the values,

$$C_D = (1/2) \times 0.7364 \times \pi \times 0.02^2 u^2 = (4/3) \times \pi \times 0.02^3 \times 9.81 \times 940$$

$$\therefore \quad C_D u^2 = 667.85$$

C_D depends on Reynolds number which cannot be calculated without the value of velocity. Looking at the graph for C_D for spheres, the value is about 0.45 for $Re = 10^3$ to 5×10^5. Substituting this value, **u = 38.52 m/s or 138.7 kmph.**

$$\text{Re} = 38.52 \times 0.04/1.628 \times 10^{-5} = 0.94 \times 10^5.$$

Hence the assumed value of C_D is acceptable.

Problem 6.16 *A stirrer is constructed as shown in Fig. P. 6.16. The dimensions are indicated in the figure. The stirrer speed is 90 rpm.* ***Determine the torque*** *on the shaft and also* ***the power required****. Assume the vessel is large. Neglect the drag on the rod and the shaft. Density of the fluid is 1025 kg/m*^{3}.

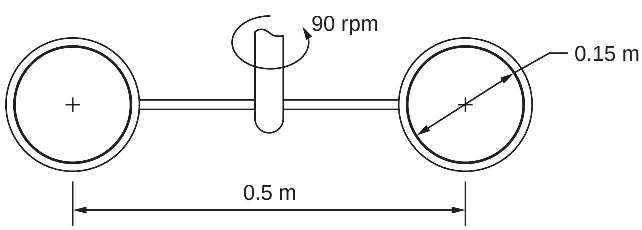


Figure P. 6.16 Stirrer details

For circular plate $C_D = 1.17$

Linear speed of the disk $= \dfrac{\pi DN}{60} = \dfrac{\pi \times 0.5 \times 90}{60} = 4.7124$ m/s

$$C_D = F_D/(1/2)\,\rho\, AV^2,\ A = \pi \times 0.15^2/4$$

$\therefore$ $$F_D = 1.17 \times (1/2) \times 1025 \times (\pi \times 0.15^2/4) \times 4.7124^2 = 235.31 \text{ N}$$

Torque = Force × torque arm = 235.30 × 0.5 = **117.65 Nm.**

Power = 2π NT/60 = (2π × 90 × 117.65)/60 = **1109 W**

Problem 6.17 *An anemometer has hemispherical cups of 80 mm dia with an arm distance from the post to center of 130 mm. If due ot fiction, the cups starts rotating at a wind speed of 3 m/s. **Determine the starting torque**. Consider density of air as 1.23 kg/m³.*

The coefficient of drag when the cup faces the wind is 1.42. The coefficient of drag on the back = 0.38.

$\therefore$ Net coefficient = 1.42 – 0.38 = 1.04

$\therefore$ Force = $C_D A\rho\, V^2/2$, Torque = Force × torque arm, Substituting

Starting torque $= 1.04 \times \pi \times \dfrac{0.08^2}{4} \times \dfrac{1.23}{2} \times 3^2 \times 0.13 = \mathbf{3.76 \times 10^{-3}\ Nm}$

Problem 6.18 ***Determine the wind force*** *on the antenna shown in Figure P. 10.18. All the components face the wind blowing at 100 kmph. ρ = 1.205 kg/m³, kinematic viscosity is 15.06 × 10⁻⁶ m²/s.*

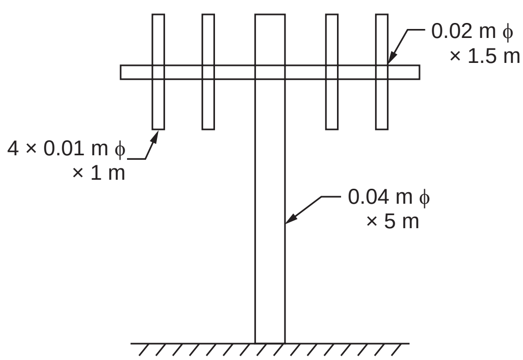


Figure P. 6.18 Antenna details

Velocity of wind = 100000/3600 = 27.78 m/s

The value of Reynolds number is given by

Re = $27.78 \times 0.04/15.06 \times 10^{-6} = 7.38 \times 10^4$ for 40 mm rod and 1.84×10^4 for 10 mm rod. At this value C_D is about 1.4 for both cases.

$$\mathbf{F_D} = 1.4 \times \frac{1.205}{2} \times 27.78^2\,[(5 \times 0.04) + (1.5 \times 0.02) + (4 \times 0.01)]$$

$$= \mathbf{175.7\ N}$$

Problem 6.19 *The total mass of an aircraft is 70000 kg. The wing area is 160 m* ^{2}*. If the craft travels at 600 kmph,* ***determine the lift coefficient****. Neglect the compressibility effect. Air density at the flight conditions is 0.85 kg/m*^{3}*.*

The lift force should be equal the weight at steady flight. Lift force F_L is given by

$$F_L = C_L\,A(1/2)\,\rho\,V^2, \text{ flight speed, } V = 600000/3600 = 166.7 \text{ m/s}$$

$$70000 \times 9.81 = C_L \times 160 \times (1/2) \times 0.85 \times 166.7^2 \quad \therefore\ \mathbf{C_L = 0.3635}.$$

Problem 6.20 *In championship tennis, balls are hit at speeds exceeding 100 kmph and good amount of spin.* ***Calculate the aerodynamic lift on ball and radius of curvature of path*** *in the vertical plane, when the ball is hit at a speed of 108 kmph and a top spin of 8000 rpm. The ball diameter is 0.064 m and mass is 0.057 kg. For air density = 1.165 kg/m*^{3} *and kinematic viscosity is 16 ×* 10^{-6} *m*^{2}*/s.*

The lift force depends on the spin ratio and Reynolds number. Top spin causes downward force. Spin ratio = $\omega D/2u$.

$$u = 108000/3600 = 30 \text{ m/s}, \ \omega = 8000 \times 2\pi/60 = 837.76 \text{ radians/s}$$

$$\therefore \quad \text{Spin ratio} = (837.76 \times 0.064/2 \times 30) = 0.8936$$

$$Re = 30 \times 0.064/16 \times 10^{-6} = 1.2 \times 10^5$$

By interpolation in Fig. 10.3.4, page 341 C_L is read as 0.25

$$\therefore \quad \textbf{Lift force} = 0.25 \times \pi \times \frac{0.064^2}{4} \times \frac{1}{2} \times 1.165 \times 30^2 = \mathbf{0.4216\ N},$$

This force acts downwards due to top spin.

$$\text{gravity force} = 0.057 \times 9.81 = 0.5592 \text{ N} \quad \therefore \text{ Total force} = 0.9808 \text{ N}$$

Equating it to the z directional acceleration

$$F = mu^2/R \text{ where } R \text{ is the radius of the path in the vertical plane.}$$

$$\mathbf{R} = 0.057 \times 30^2/0.9808 = \mathbf{52.3\ m}$$

In case only gravity force acts, then $R = 0.057 \times 30^2/0.5592 = 91.7$ m

The ball comes down sharply due to the top spin.

Problem 6.21 *A table tennis ball of mass 2.5 grams and a diameter of 38 mm is hit with a velocity of 12 m/s, with a back spin ω.* ***Determine the value of back spin for the ball to travel in a horizontal path****, not dropping due to gravity. Density and kinematic viscosity of air are 1.165 kg/m*^{3} *and 16 ×* 10^{-6} *m*^{2}*/s.*

For this situation, the force due to gravity should equal the lift force. *i.e.,*

$$mg = C_L(1/2)\,\rho\,u^2 A$$

$$\frac{2.5}{1000} \times 9.81 = C_L \times \frac{1}{2} \times 1.165 \times 12^2 \times \pi \times \frac{0.038^2}{4} \quad \therefore C_L = 0.2578$$

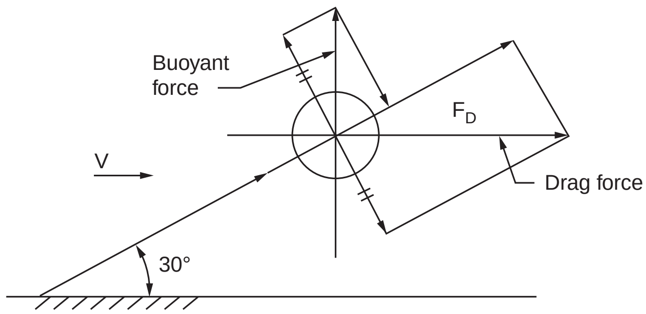
From the graph for C_L vs spin ratio, (Fig. 10.3.4) the value of spin ratio is read as 0.91

$$\frac{\omega D}{2u} = 0.91, \quad \therefore \quad \boldsymbol{\omega} = \mathbf{574.7\ rad/s\ or\ 5488\ rpm}$$

If the velocity is more, for this spin the ball will rise. If the velocity is less then the ball will travel in an arc.

Problem 6.22 *A cork ball 0.3 m diameter with specific gravity 0.21 is tied on the bed of a river. At a certain time it rests at 30° to the horizontal due to the flow.* ***Determine the velocity of flow.***

The forces on the cork ball are shown in Fig. P. 10.22.



At equilibrium the components along the rope (at 30° to the horizontal) is taken up by the rope. The components perpendicular to this line should balance.

$\therefore \quad F_D \cos 60 = F_b \cos 30$

F_b = Buoyant force = difference in density × volume × g

$$= 790 \times \frac{4}{3}\pi \times 0.15^3 \times 9.81 \text{ N} = 109.56 \text{ N}$$

$\therefore \quad F_b \cos 30 = 94.88$ N. $\quad \therefore F_D = 94.88/\cos 60 = 189.66$ N

$F_D = C_D\,(1/2)\,\rho\, AV^2$, $\boldsymbol{C_D}$ **= 0.45 for sphere**

$$189.66 = \frac{0.45}{2} \times 1000 \times \pi \times \frac{0.3^2}{4}V^2, \text{ Solving } \mathbf{V = 3.45\ m/s}$$

Reynolds number $= \dfrac{3.45 \times 0.3}{1.06 \times 10^{-6}} = 0.98 \times 10^6$

For this value $\mathbf{C_D = 0.2}$ Corresponding **V = 5.18 m/s**

Further iteration is necessary as the new value of Re = 1.47×10^6 and $C_D = 0.35$.

Problem 6.23 *Air flows in a square duct of side 0.6 m with a velocity of 3 m/s. The displacement thickness in meter is given by* $\delta_d = 0.0039\, x^{0.5}$ *where x is the distance along the flow.* ***Determine the velocity outside the boundary layer at a distance of 30 m. Density of air*** *= 1.2 kg/m*^{3}.

The flow with boundary layer can be taken as flow at the free stream velocity with the boundary moved by a distance equal to the displacement thickness.

At 30 m, displacement thickness is $\delta_d = 0.0039 \times 30^{0.5} = 0.02136$ m

The side of the square is reduced by twice this thickness.

∴ Length of side considering displacement thickness is

$$L_d = (0.6 - 2 \times 0.02136) = 0.5573 \text{ m}$$

Equating the volume flow rate $0.6 \times 0.6 \times 3 = 0.5573^2 \times V_2$

∴ $\mathbf{V_2 = 3.48 \text{ m/s}}$

The pressure drop can be calculated for the flow outside the boundary layer as

$$\Delta P = (1/2)\, \rho\, (V_2^2 - V_1^2) = (1/2) \times 1.2\, [3.48^2 - 3^2] = 1.87 \text{ N/m}^2$$

OBJECTIVE QUESTIONS

O Q. 10.1 Fill in the blanks:

1. In flow over surfaces, fluid at the surface takes on the velocity of the body as a result of _______________ condition.
2. The study of non viscous fluid flow is called _______________.
3. Equations describing the complete flow field are know as _______________ equations.
4. The effect of viscosity is important only in a thin layer adjacant to the surface called _______________.
5. The flow outside the boundary layer can be treated as _______________ flow.
6. Velocity gradient exists only in the _______________.
7. The forces which are important in the boundary layer are _______________.
8. In ideal flwo the forces that are important are _______________.
9. The pressure gradient at the surface causes _______________ on the surface.
10. Initially _______________ flow prevails in the boundary layer.

Answers

(1) no slip (2) Theoretical hydrodynamics. (3) Navier-Stokes (4) boundary layer (5) Ideal fluid (6) boundary layer (7) Inertia and viscous forces (8) pressure and inertia (9) shear stress (10) Laminar

O Q. 10.2 Fill in the blanks:

1. Mass and momentum flow in laminar boundary layer is only at the_______________ level.
2. The two methods of analysis of boundary layer flow are _______________.
3. Macroscopic mixing between layers occurs in _______________.
4. The ratio of inertia force to viscous force is called _______________ number.
5. Turbulent flow over a flat plate is generally taken to start at a Reynolds number of _______________.
6. In laminar flow viscous forces are _______________ compared to inertia forces.
7. In turbulent flow viscous forces are _______________ compared to inertia force.
8. Boundary layer separation occurs when there is an _______________ pressure gradient.
9. Lift is the component of the total force on a body immersed in a flow in the _______________ direction.
10. Drag is the component of the total force on a body immersed in a flow in the _______________ direction.