5.1 INVERSE Z-TRANSFORMS BY PARTIAL FRACTION METHOD

Z-Transform of some basic functions:	
1.	$Z\left[a^n\right]=\frac{z}{z-a}$; $Z[1]=\frac{z}{z-1}$; $Z\left[(-a)^n\right]=\frac{z}{z+a}$
2.	$Z[n]=\frac{z}{(z-1)^2}$
3.	$Z\left[\frac{1}{n}\right]=\log\left(\frac{z}{z-1}\right)$
4.	$Z\left[\frac{1}{n+1}\right]=z\log\left(\frac{z}{z-1}\right)$
5.	$Z\left[\frac{1}{n-1}\right]=\frac{1}{z}\log\left(\frac{z}{z-1}\right)$
6.	$Z\left[\frac{1}{n!}\right]=e^{\frac{1}{z}}$
7.	$Z[\cos n\theta]=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
8.	$Z[\sin n\theta]=\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
Inverse Z-Transforms: The inverse Z-transform of $Z[f(n)]=F(z)$ is defined as $f(n)=Z^{-1}[F(z)]$.	
The inverse Z-Transform of some basic functions:	
1.	$Z^{-1}\left[\frac{z}{z-1}\right]=1$; $Z^{-1}\left[\frac{z}{z+1}\right]=(-1)^n$
2.	$Z^{-1}\left[\frac{z}{z-a}\right]=a^n$; $Z^{-1}\left[\frac{z}{z+a}\right]=(-a)^n$; $Z^{-1}\left[\frac{1}{z+a}\right]=a^{n-1}$
3.	$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=(n+1)a^n$ For Eg. 1) $Z^{-1}\left[\frac{z}{(z-a)^2}\right]=(n-1+1)a^{n-1}=na^{n-1}$ 2) $Z^{-1}\left[\frac{1}{(z-a)^2}\right]=(n-2+1)a^{n-2}=(n-1)a^{n-2}$ 3) $Z^{-1}\left[\frac{z^2}{(z-1)^2}\right]=(n+1)1^n=n+1$ 4) $Z^{-1}\left[\frac{z}{(z-1)^2}\right]=(n-1+1)1^n=n$ 5) $Z^{-1}\left[\frac{1}{(z-1)^2}\right]=(n-2+1)1^n=n-1$
4.	$Z^{-1}\left[\frac{z^2}{z^2+a^2}\right]=a^n\cos\frac{n\pi}{2}$

5.	$Z^{-1}\left[\frac{z}{z^2+a^2}\right]=a^n\cos(n-1)\frac{\pi}{2}=a^n\cos\left(\frac{\pi}{2}-\frac{n\pi}{2}\right)=a^n\sin\frac{n\pi}{2}$

Finding Inverse Z-transform by method of **Partial Fractions**:

Rules of Partial Fractions:

1. Denominator containing Linear factors:

$$\frac{f(z)}{(z-a)(z-b)(z-c)...}=\frac{A}{(z-a)}+\frac{B}{(z-b)}+\frac{C}{(z-c)}+...$$

2. Denominator containing factors $(z-a)^n$:

$$\frac{f(z)}{(z-a)^n}=\frac{A}{(z-a)}+\frac{B}{(z-a)^2}+\frac{C}{(z-a)^3}+...+\frac{D}{(z-a)^n}$$

3. Denominator contains a quadratic factor of the form az^2+bz+c (where a,b,c are constants):

$$\frac{f(z)}{az^2+bz+c}=\frac{A}{az^2+bz+c}+\frac{Bz}{az^2+bz+c}$$

(Or) $\frac{f(z)}{az^2+bz+c}=\frac{Az+B}{az^2+bz+c}$

1. **Find** $Z^{-1}\left[\frac{z}{(z+1)(z-1)^2}\right]$ **using the method partial fraction.**

Solution:

$$F(z)=\frac{z}{(z+1)(z-1)^2}$$

$$\frac{F(z)}{z}=\frac{1}{(z+1)(z-1)^2}\quad -----(1)$$

Now,

$$\frac{1}{(z+1)(z-1)^2}=\frac{A}{z+1}+\frac{B}{z-1}+\frac{C}{(z-1)^2}$$

$$1=A(z-1)^2+B(z+1)(z-1)+C(z+1)$$

Put $z=1\Rightarrow 1=2C\quad\Rightarrow \boxed{C=\frac{1}{2}}$

Put $z=-1,\ \Rightarrow\ 1=4A\ \Rightarrow\ \boxed{A=\frac{1}{4}}$

Put $z=0\ \Rightarrow\ 1=A-B+C\ \Rightarrow B=\frac{1}{4}+\frac{1}{2}-1\ \Rightarrow B=\frac{1+2-4}{4}\ \Rightarrow \boxed{B=\frac{-1}{4}}$

$$\frac{1}{(z+1)(z-1)^2}=\frac{\frac{1}{4}}{z+1}+\frac{\frac{-1}{4}}{z-1}+\frac{\frac{1}{2}}{(z-1)^2}$$

$$(1)\ \Rightarrow F(z)=\frac{1}{4}\frac{z}{z+1}-\frac{1}{4}\frac{z}{z-1}+\frac{1}{2}\frac{z}{(z-1)^2}$$

Taking Z^{-1} on both sides

$$(1)\ \Rightarrow Z^{-1}\left[F(z)\right]=\frac{1}{4}Z^{-1}\left[\frac{z}{z+1}\right]-\frac{1}{4}Z^{-1}\left[\frac{z}{z-1}\right]+\frac{1}{2}Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$\boxed{f(n)=\frac{1}{4}(-1)^n-\frac{1}{4}(1)+\frac{1}{2}n}$$

2\. **Find** $Z^{-1}\left[\dfrac{z^{-2}}{\left(1+z^{-1}\right)^2\left(1-z^{-1}\right)}\right]$.

Solution:

$$F(z)=\frac{z^{-2}}{\left(1+z^{-1}\right)^2\left(1-z^{-1}\right)}=\frac{1}{\cancel{z^2}\dfrac{(z+1)^2}{\cancel{z^2}}\left(\dfrac{z-1}{z}\right)}$$

$$F(z)=\frac{z}{(z+1)^2(z-1)}$$

$$\frac{F(z)}{z}=\frac{1}{(z+1)^2(z-1)} \quad -----(1)$$

$$\frac{1}{(z-1)(z+1)^2}=\frac{A}{z-1}+\frac{B}{z+1}+\frac{C}{(z+1)^2}$$

$$1=A(z+1)^2+B(z-1)(z+1)+C(z-1)$$

Put $z=1,\quad 1=4A \quad\Rightarrow \boxed{A=\frac{1}{4}}$

Put $z=-1,\ \Rightarrow\ 1=-2c\ \Rightarrow \boxed{c=-\frac{1}{2}}$

Equating co-efficients of $z^2 \quad\Rightarrow\quad 0=A+B\ \Rightarrow \boxed{B=-\frac{1}{4}}$

$$(1)\ \Rightarrow \frac{F(z)}{z}=\frac{1}{4}\frac{1}{z-1}+\frac{-1}{4}\frac{1}{z+1}-\frac{1}{2}\frac{1}{(z+1)^2}$$

$$(1)\ \Rightarrow Z^{-1}\left[F(z)\right]=\frac{1}{4}Z^{-1}\left[\frac{z}{z-1}\right]-\frac{1}{4}Z^{-1}\left[\frac{z}{z+1}\right]-\frac{1}{2}Z^{-1}\left[\frac{z}{(z+1)^2}\right]$$

$$f(n)=\frac{1}{4}(1)^n-\frac{1}{4}(-1)^n+\frac{1}{2}n(-1)^n$$

$$\boxed{f(n)=\frac{1}{4}-\frac{1}{4}(-1)^n+\frac{1}{2}n(-1)^n}$$

3\. **Find** $Z^{-1}\left[\dfrac{z^{-2}}{\left(1-z^{-1}\right)\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}\right]$.

Solution:

$$F(z)=\left[\frac{z^{-2}}{\left(1-z^{-1}\right)\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}\right]=\frac{\dfrac{1}{z^2}}{\left(1-\dfrac{1}{z}\right)\left(1-\dfrac{2}{z}\right)\left(1-\dfrac{3}{z}\right)}$$

$$=\frac{1}{\cancel{z^2}\left(\dfrac{z-1}{\cancel{z}}\right)\left(\dfrac{z-2}{\cancel{z}}\right)\left(\dfrac{z-3}{z}\right)}$$

$$F(z)=\frac{z}{(z-1)(z-2)(z-3)}$$

$$\frac{F(z)}{z}=\frac{1}{(z-1)(z-2)(z-3)} \quad -----(1)$$

Now by Partial Fraction,

$$\frac{1}{(z-1)(z-2)(z-3)}=\frac{A}{z-1}+\frac{B}{z-2}+\frac{C}{z-3}$$

$$1=A\left(z-2\right)\left(z-3\right)+B\left(z-1\right)\left(z-3\right)+C\left(z-1\right)\left(z-2\right)$$

Put $z=2, \quad \Rightarrow \; 1=-B\Rightarrow \boxed{B=-1}$

Put $z=1, \quad \Rightarrow \; 1=2A\Rightarrow \boxed{A=\frac{1}{2}}$

Put $z=3, \quad \Rightarrow \; 1=2C\Rightarrow \boxed{C=\frac{1}{2}}$

$$(1)\Rightarrow F(z)=\frac{1}{2}\frac{z}{z-1}-\frac{z}{z-2}+\frac{1}{2}\frac{z}{z-3}$$

$$(1)\Rightarrow Z^{-1}\left[F(z)\right]=\frac{1}{2}Z^{-1}\left[\frac{z}{z-1}\right]-Z^{-1}\left[\frac{z}{z-2}\right]+\frac{1}{2}Z^{-1}\left[\frac{z}{z-3}\right]$$

$$f(n)=\frac{1}{2}(1)^n-(2)^n+\frac{1}{2}(3)^n$$

$$\boxed{f(n)=\frac{1}{2}-2^n+\frac{1}{2}3^n}$$

4. **Find the Z-transform of** $\frac{z^2+z}{(z-1)(z^2+1)}$ **using partial fraction.**

Solution:

$$F(z)=\frac{z^2+z}{(z-1)(z^2+1)}$$

$$\frac{F(z)}{z}=\frac{z+1}{(z-1)(z^2+1)}$$

$$\frac{z+1}{(z-1)(z^2+1)}=\frac{A}{(z-1)}+\frac{B}{(z^2+1)}+\frac{Cz}{(z^2+1)}$$

$$z+1=A(z^2+1)+B(z-1)+Cz(z-1)$$

Put $z=1, \quad \Rightarrow \; 2=2A\Rightarrow \boxed{A=1}$

Equating co-efficients of $z^2 \;\Rightarrow 0=A+C \;\Rightarrow \boxed{C=-1}$

Put $z=0, \quad \Rightarrow \; 1=A-B\Rightarrow B=A-1=1-1=0\,\boxed{B=0}$

$$\frac{F(z)}{z}=\frac{1}{(z-1)}+\frac{0}{(z^2+1)}+\frac{-z}{(z^2+1)}$$

$$F(z)=\frac{z}{(z-1)}-\frac{z^2}{(z^2+1)}$$

Put Z^{-1} on both sides

$$Z^{-1}\left[F(z)\right]=Z^{-1}\left[\frac{z}{z-1}\right]-Z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$\boxed{f(n)=1-\cos\frac{n\pi}{2}} \qquad \because Z^{-1}\left[\frac{z^2}{z^2+a^2}\right]=\cos\frac{n\pi}{2}$$