ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MATHEMATICS

LECTURE NOTES ON

BA4201 / QUANTITATIVE TECHNIQUES FOR DECISION MAKING

UNIT V : QUEUING THEORY

BASIC CONCEPTS OF QUEUING MODELS

Introduction: Queuing theory is the mathematical study of waiting lines or queues. The theory enables mathematical analysis of several related process. Queues are inevitable in all areas of business. These are formed to get service at that time when demand for a service is more then the capacity of service facility. In queues customers arrive at a greater rate than service facility.

Characteristics of Queuing theory:

The study of queuing theory involves the study of different operating characteristics.
Some of the more commonly considered characteristics are discussed below

- **System Length:** the average number of customers in the queue waiting to get service.
- **System Length:** the average numbers of customers in the system those waiting to be and those being serviced**.**
- **Waiting time in the queue:** The average time the customer has to wait in the queue to get serviced
- **Total time in the system:** the average time that a customer spends in the system from entry in the queue to completion of service**.**
- **Server Idle Time:** the relative frequency with which the service system is idle.

Basic Elements of Queuing Model:

Elements of queuing system may also be termed as the lements of the queuing theory. The following atr a few important elements.

- **Arrival Process**
    - According to source: The arrivals cam be from a population that is infinite or they cam be from finite population
    - According to numbers: Customers may arrive for service in single or in group.
    - According to time: Arrival cam be at regular interval of time or at random.
- **Service System**
    - Single server facility: there may be only one service facility available and it is defined as a single channel service system.(doctor)
    - Multiple, parallel facilities with single queue: there may be more than one server and parallel implies that each server provides the same type of facility. (railway booking counter)
    - Multiple, parallel facilities with multiple queues: This is different from the earlier that each of the servers have different queues
    - Service facilities in a series: In this customer enters the first station and gets a portion of service and then again moves on to the next station
- **Queue Structure**
    - First come first served
    - Last come first served
    - Service in random order
    - Priority service

Customer Behavior in Queue

- Balking: A customer may not like to wait in a queue due to lack of time or space or otherwise.
- Reneging: A customer may leave the queue due to impatience

- Collusion: some customers may collaborate and one them may join the queue .
- Jockeying: if there are more than one queue then one customer may leave one queue and join the queue.

Model 1: Poisson-exponential single server model – infinite population

Assumptions:

- Arrivals are Poisson with a mean arrival rate of, say λ
- Service time is exponential, rate being μ
- Source population is infinite
- Customer service on first come first served basis
- Single service station

For the system to be workable, $\lambda \leq \mu$

Model 2: Poisson-exponential single server model – finite population

Has same assumptions as model 1, except that population is finite

- Model 3: Poisson-exponential multiple server model – infinite population

Assumptions

- Arrival of customers follows Poisson law, mean rate λ
- Service time has exponential distribution, mean service rate μ
- There are K service stations
- A single waiting line is formed
- Source population is infinite
- Service on a first-come-first-served basis
- Arrival rate is smaller than combined service rate of all service facilities

Model: 1 Operating Characteristics

a) Queue length
    - average number of customers in queue waiting to get service
b) System length
    - average number of customers in the system

c) Waiting time in queue
   - average waiting time of a customer to get service

d) Total time in system
   - average time a customer spends in the system

e) Server idle time
   - relative frequency with which system is idle

- *Measurement parameters*
- λ= mean number of arrivals per time period (eg. Per hour)
- μ = mean number of customers served per time period
- Probability of system being busy/traffic intensity

$\rho = \lambda / \mu$

- Average waiting time system $Ws = 1/(\mu - \lambda)$
- Average waiting time in queue

$Wq = \lambda / \mu(\mu - \lambda)$

- Average number of customers in the system

$Ls = \lambda / (\mu - \lambda)$

- Average number of customers in the queue

$Lq = \lambda 2 / \mu(\mu - \lambda)$

- Probability of an empty facility/system being idle

$P(0) = 1 - P(w)$

- Probability of being in the system longer than time (t)

$P(T>t) = e -(\mu - \lambda)t$

Probability of customers not exceeding k in the system

$P(n. \geq k) = \rho k$

$P(n>k) = \rho(k+1)$

Probability of exactly N customers in the system

$P(N) = \rho N (1-\rho)$

Example 1. Customers arrive at a booking office window, being manned by a single individual rate of 25per hour. Time required to serve a customer has exponential distribution with a mean of 120 seconds. Find the mean waiting time of a customer in the queue

Solution: Given λ =25 customer/hr

μ = 30 customer/hr

Mean waiting time in Queue = $\lambda / \mu (\mu - \lambda)$

= 25/30 (30-25)

= 0.167 hr

Mean waiting time in system =$1/ \mu - \lambda$

=1/30-25

=0.2 hr

Problem 2: A repairman is to be hired to repair machines which breakdown at a n average rate of 6 per hour. The breakdowns follow Poisson distribution. The non-production time of a machine is considered to cost Rs. 20 per hour. Two repairmen Mr. X and Mr.Y have been interviewed for this purpose. Mr. X charges Rs.10 per hour and he service breakdown machines at the rate of 8 per hour. Mr. Y demands Rs.14 per hour and he services at an average of 12 per hour. Which repairman should be hired? (Assume 8 hours shift per day)

solution: Given λ =6 /hr

μ_x=8/hr

μ_y=12/hr

Given no of machine cost at idle = 20Rs/hr

No of machine in X $Ls_x = \lambda / \mu_x - \lambda$

=6/8-6

=3 machines

Total no of machines =3*8=24 machines

Total cost=hiring charges of x + cost of idle machine

=10*8 + 24*20

=Rs.560

No of machine in y $Ls_y = \lambda / \mu_y - \lambda$

=6/12-6

=1

Total no of machine = 1*8=8 machines

Total cost= hiring charges of y + cost of idle machine

= 14*8 + 20*8

= Rs.272

We chose Mr.Y since cost is lower than Mr. X

Problem 3: A warehouse has only one loading dock manned by a three person crew. Trucks arrive at the loading dock at an average rate of 4 trucks per hour and the arrival rate is Poisson distributed. The loading of a truck takes 10 minutes on an average and can be assumed to be exponentially distributed . The operating cost of a truck is Rs.20 per hour and the members of the crew are paid @ Rs.6 each per hour. Would you advise the truck owner to add another crew of three persons?

Solutions: 1) λ=4 truck/ hr

μ=10min=60/10=6truck/hr

No of trucks in system,

$$Ls= \lambda / (\mu - \lambda)$$

=4/6-4

=2 trucks

Total cost = cost of maintaining trucks + cost of crew

=20*20 + 3*6

=40 + 18

= Rs.58

2) λ =4trucks /hr

If we double the crew, μ= 12 trucks/ hr

No of trucks in system,

$$Ls= \lambda / (\mu - \lambda)$$

=4/12-4

=0.5 trucks

Total cost = cost of maintaining trucks + cost of crew

=0.5*20 + 6*6

=10 + 36

=Rs.46

We will advice the owner to add 3 persons for loading cost the total cost for 3 extra persons is less than previous.

Problem 4 At a service counter of fast-food joint, the customers arrive at the average interval of six minutes whereas the counter clerk takes on an average 5 minutes for preparation of bill and delivery of the item. Calculate the following

a. counter utilization level

b. average waiting time of th4e customers at the fast food joint

c. Expected average waiting time in the line

d. Average number of customers in the service counter area

e. average number of customer in the line

f. probability that the counter clerk is idle

g. Probability of finding the clerk busy

h. chances that customer is required to wait more than 30 minutes in the system

i. probability of having four customer in the system

J) probability of finding more than 3 customer in the system

Solutions: Given $\lambda = 60/10 = 10$ customer/hr

$\mu = 12$ customer/hr

a) $\rho = \lambda / \mu = 10/12 = 0.833$
b) $Ws = 1/ \mu - \lambda = 1/12-10 = 0.5$ hr
c) $Wq = \lambda / \mu (\mu - \lambda) = 10/12(12-10) = 0.416$ hr
d) $Ls = \lambda / (\mu - \lambda) = 10/12-10 = 5$ customers
e) $Lq = \lambda^2 / \mu (\mu - \lambda) = 10^2 /12(12-10) = 4.167$ customers
f) $1- \rho = 1- \lambda / \mu = 1- 10/12 = 0.167$
g) $\rho = \lambda / \mu = 10/12 = 0.833$
h) chances of probability that customer wait more than 30min = 30/60 = 0.5 hrs

$P (T>t) = e^{-(\mu - \lambda) t}$

$P (T>0.5) = e^{-(12-10) 0.5} = 0.368$

i) P (N) = ρN (1-ρ)

P (4) = $\rho 4$ (1-ρ) = $(0.833)^4(1-0.833)$ = 0.0806

j) P (n>k) = ρ (k+1)

P (n>3) = ρ (3+1) = $(\lambda / \mu)^4 = (10/12)^4$ = 0.474

Problem 5: Customers arrive at a one-window drive-in bank according to a Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The space in front of the window including that for the serviced car accommodate a maximum of 3 cars. Other cars can wait outside the space. Calculate

- A) what is the probability that an arriving customer can drive directly to the space in front of the window.
- B) what is the probability that an arriving customer will have to wait outside the indicated space
- C) How long is arriving customer expected to wait before stating the service.
- D) How many spaces should be provided in front of the window so that all the arriving customers can wait in front of the window at least 20% of the time.

solution: Given λ = 10 /hr

μ = 5min = 60/5 = 12customer/hr

$\rho = \lambda / \mu$ = 10/12 = 0.83

We know that P (N) = ρN (1-ρ)

a) P = $P_o + P_1 + P_2$

= $\rho 0$ (1-ρ) + $\rho 1$ (1-ρ) + $\rho 2$ (1-ρ)

= (1- ρ) (1+ ρ + $\rho 2$)

= (1- 0.83) (1+ 0.83 + 0.83^2)

= 0.428

b) P = 1 – ($P_o + P_1 + P_2 + P_3$)

= 1 – [0.42 + $\rho 3$ (1-ρ)]

= 1 – [0.42 + 0.833 (1-0.83)]

= 0.482

c) Wq = $\lambda / \mu (\mu - \lambda)$ = 10/12(12-10) = 0.416 hr

d) $P_o = \rho 0 (1-\rho)$ 1-10/12 = 0.16

e) $P_1 = \rho 1 (1-\rho)$ = 10/12(1-10/12) = 0.14

Probability that there will be no or one car in the space which cover = $P_o + P_1$ = 0.30 = 30%, hence there should at least 1 car space for waiting at the window space at least 20% of the time.

Problem 6

Customers arrive at the first class ticket counter of a theatre at a rate of 12 per hours. There is one clerk serving the customers at a rate of 30 per hour. Assuming the conditions for use of the single channel queuing model, evaluate

a) The probability that there is no customer at the counter (i.e. that the system is idle)
b) The probability that there are more than 20 customers at the counter
c) The probability that there is no customer waiting to be served
d) The probability that a customer is being served and no body is waiting.

Solution: Given λ = 12 customers /hr

μ = 30 customers/hr

1) Ideal =1- ρ = 1- λ / μ = 1- 12/30 = 0.6

2) At least 3 customers at counter

$$P(n>k) = \rho (k+1) = \rho (3+1) = (12/30)^4 = 0.025$$

3) Probability that no customers waiting to be served

P (at least 1 customer at the counter)

$= P_o + P_1$

$= \rho 0 (1-\rho) + \rho 1 (1-\rho) = (1-\rho)(1+\rho) = (1-\rho^2)$

$= (1-(12/30)^2 = 0.84$

4) Probability that a customers being served and nobody waiting

$= P_1 = \rho 1 (1-\rho)$ = 12/30(1-12/30) = 0.24

Problem 7: A Telephone exchange has two long distance operators. The telephone company finds that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes.

(1) What is the probability that a subscriber will have to wait for his long distance

call during the peak hours of the day?

(2) If the subscribers will wait and are serviced in turn, what is the expected waiting time?

Soln.

$$\lambda = \frac{15}{60} = \frac{1}{4}\ ,\ \mu = \frac{1}{5},\ s = 2 \text{ and } \rho = \frac{\lambda}{s\mu} = \frac{5}{8}$$

$$\Pr ob(w > 0) = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)} \times P_0, \text{ where } P_0 \text{ is}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1} \Rightarrow \left[\sum_{n=0}^{1} \frac{\left(\frac{5}{4}\right)^n}{n!} + \frac{\left(\frac{5}{4}\right)^2}{2!\left(1-\frac{5}{8}\right)}\right]^{-1} \Rightarrow \left[\sum_{n=0}^{2} \frac{\left(\frac{5}{3}\right)^n}{n!} + \frac{\left(\frac{5}{3}\right)^3}{6\times 4}\right]^{-1}$$

$$\Rightarrow \left[1 + \frac{5}{3} + \frac{25}{9} \times \frac{1}{2} + \frac{75}{27} \times \frac{9}{24}\right]^{-1} = \left[\frac{13}{3}\right]^{-1} = \frac{3}{13}$$

$$(1)\ \Pr ob(w > 0) = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)} \times P_0 \Rightarrow \frac{\left(\frac{5}{4}\right)^2 \frac{3}{13}}{2!\left(1-\frac{5}{8}\right)} = \frac{25}{52} = 0.48$$

$$(2)\ W_q = \frac{L_q}{\lambda} = \frac{\rho(s\rho)^2}{s!\,(1-\rho)^2} P_0 \Rightarrow \frac{4\times\frac{5}{8}\left(\frac{5}{4}\right)^2}{2!\left(1-\frac{5}{8}\right)} \times \frac{3}{13} = \frac{125}{39} = 3.2 \text{ minutes}$$

Problem 8: People arrive at a theatre ticket booth in Poisson distributed arrival rate of 25 per hour. Service time is constant at 2 minutes. Calculate (1) the mean number in waiting line (2) the mean waiting time (3) the utilization factor.

Soln.

$$\lambda = 25 \text{ per hour},\ \mu = \frac{1}{2} \times 60 = 30 \text{ per hour},\ \rho = \frac{\lambda}{\mu} = \frac{25}{30} = 0.833$$

$$(1)\ \text{Length of queue } L_q = \frac{\rho^2}{1-\rho} = 4\ (appr.)$$

$$(2)\ \text{Mean waiting time} = \frac{L_q}{\lambda} = 9.6 \text{ minutes}$$

$$(3)\ \text{Utilisation factor} = \rho = 0.833$$

Problem 9 : In a public telephone booth the arrivals are on the average 15 per hour. A call on the average takes 3 minutes. If there is just one phone, find (1) expected number of callers in the booth at any time (2) the proportion of the time the booth is expected to be idle?

Soln.

$$\lambda = 15 \ per \ hour, \ \mu = \frac{1}{3} \times 60 = 20 \ per \ hour, \ \rho = \frac{\lambda}{\mu} = \frac{15}{20} = \frac{3}{4}$$

$$(1) \ Expected \ \ Length \ of \ the \ non-empty \ Queue = \frac{\mu}{\mu - \lambda} = \frac{20}{20-15} = 4$$

$$(2) \ The \ booth \ \ for \ \exp ected \ \ idle \ = 1 - \rho = 1 - \frac{3}{4} = \frac{1}{4} hrs$$

Problem 10: A petrol station has two pumps. The service time follows the exponential distribution with mean 4 minutes and cars arrive for service in a Poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service. What proportion of time the pump remains idle?

Soln.

Here, $\lambda = 10 \ per \ hour, \ \mu = \frac{1}{4} \min ute = \frac{60}{4} = 15 \ per \ hour$ & s = 2

The proportion of time, the pumps remain busy is given by $\rho = \frac{\lambda}{s\mu} = \frac{10}{2 \times 15} = 0.33$

Therefore the proportion of time,

the pump remains idle = $1 - \rho = \frac{2}{3} \Rightarrow \% \ of \ \ period \ is \ 67\%$.

$$P(w > 0) = \frac{P_s}{1-\rho}, \text{ where } P_s = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \times P_0 \text{ , where } P_0 \text{ is,}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} \right]^{-1} \Rightarrow \left[\sum_{n=0}^{1} \frac{\left(\frac{10}{30} \times 2\right)^n}{n!} + \frac{\left(\frac{10}{30} \times 2\right)^2}{3!\left(1 - \frac{10}{30}\right)} \right]^{-1}$$

$$\Rightarrow \left[\sum_{n=0}^{2} \frac{\left(\frac{2}{3}\right)^n}{n!} + \frac{\left(\frac{2}{3}\right)^2}{6 \times 0.67} \right]^{-1} \quad \Rightarrow \frac{1}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{2}$$

$$P_s = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \times P_0 \Rightarrow \frac{\left(\frac{2}{3}\right)^2 \frac{1}{2}}{2!} = \frac{1}{9}$$

Therefore $P(w > 0) = \frac{P_s}{1-\rho} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{1}{9} \times \frac{3}{2} = 0.167\ (app.)$

Problem 11

Cars arrive at a petrol pump, having one petrol unit, in poisson fashion with an average of 10 cars per hour. The service time is distributed exponentially with a mean of 3 minutes. Find (1) average number of cars in the system. (2) average waiting time in the queue (3) average queue length (4) the probability that the number of cars in the system is 2.

Soln. Mean arrival rate, $\lambda = 10\ per\ hour$, Mean service rate , $\mu = \frac{1}{3} \times 60 = 20\ per\ hr$

$$\rho = \frac{\lambda}{\mu} = \frac{10}{20} = \frac{1}{2}$$

(1) Average number of cars in the system, $L_S = \frac{\rho}{1-\rho} = 1\ car$

(2) Average Queue length, $L_q = \frac{\rho^2}{1-\rho} = 0.5\ car$

(3) Average waiting time in the queue, $= \frac{L_q}{\lambda} = 0.05\ hour$

(4) Probability of n units in the system, $P_n = P_n(1-\rho) \Rightarrow when\ n = 2,\ P_2 = \frac{1}{8}$

Problem 12:

A general Insurance company has three claim adjusters in its branch office. People with claims against the company are found to arrive in Poisson fashion at an average rate of 20 per 8 hour day. The amount of time that an adjuster spends with a claimant is found to have negative exponential distribution with mean service time 40 minutes. Claimants are processed in the order of their appearance.

(i) How many hours a week can an adjuster expect to spend with claimants?

(ii) How much time, on the average, does claimant spend in the branch office?

Soln.

Here $\lambda = \frac{20}{8} = \frac{5}{2}$ *arrival per hour*, $\mu = \frac{1}{4}$ *service per minute* $= \frac{3}{2}$ *per hour* & s = 3

The service is busy is given by $\rho = \frac{\lambda}{s\mu} = \frac{5}{9}$

For 5 working days, (5 x 8 = 40 hours)

Expected weekly time an adjuster spends with claimants = $\frac{5}{9} \times 40 = 22.2$ *hours*.

The average time an adjuster spends in the system is $W_s = \frac{L_s}{\lambda}$

Where $L_s = L_q + \frac{\lambda}{\mu}$ *and* $L_q = \frac{P_s\,\rho}{(1-\rho)^2}$ *and* $P_s = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \times P_0$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)}\right]^{-1} \Rightarrow \left[\sum_{n=0}^{2} \frac{\left(\frac{5}{9}\times 3\right)^n}{n!} + \frac{\left(\frac{5}{9}\times 3\right)^3}{3!\left(1-\frac{5}{9}\right)}\right]^{-1} \Rightarrow \left[\sum_{n=0}^{2} \frac{\left(\frac{5}{3}\right)^n}{n!} + \frac{\left(\frac{5}{3}\right)^3}{6\times 4}\right]^{-1}$$

$$\Rightarrow \left[1 + \frac{5}{3} + \frac{25}{9} \times \frac{1}{2} + \frac{75}{27} \times \frac{9}{24}\right]^{-1} = [5.095]^{-1} = 0.1962$$

$$\therefore\ P_s = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \times P_0 \Rightarrow \frac{\left(\frac{5}{3}\right)^3}{3!} \times 0.1962 = 0.0908$$

$$L_s = \frac{P_s\,\rho}{(1-\rho)^2} + \frac{\lambda}{\mu} \Rightarrow \frac{0.0908 \times \frac{5}{9}}{\left(1-\frac{5}{9}\right)^2} + \frac{5}{3} = 1.921$$

Therefore $W_s = \frac{L_s}{\lambda} = \frac{1.921 \times 2}{5} = 0.7684 \approx 46$ *minutes*