SUBSPACES

Definition :

Let V be a vector space and U be a non-empty subset of V. If U is a vector space under the operation of addition and scalar multiplication of V, then it is said to be a subspace of V.

Note:

(i) {0} and V itself are called trivial subspaces.

(ii) All other vector subspace of V are called non-trivial subspaces.

Note :

(i) A non-empty subset U of a vector space V over F is called subspace of V, if $u + v \in U$ and $\alpha u \in U$ for all $u, v \in U$ and $\alpha \in F$ or simply

$$\alpha u + \beta v \in U \text{ and } \alpha, \beta \in F$$

(ii) {0} is a subspace of V called zero subspace.

(iii) V is a subspace of its own.

(iv) {0} and V are called trivial subspace (or) improper subspaces.

(v) Any subspace other than{0} and V are called proper subspaces of V (or) non-trivial subspaces.

(vi) The vectors lying on a line L through the origin R^2 are subspaces of the vector space.

(vii) A non-empty subset U of vector space V is a subspace iff $u + \alpha v \in U$ for any $v \in U$ and $\alpha \in F$.

Theorem : 1.

Let w_1 and w_2 be two subspaces of vector space V over F. Then $w_1 \cap w_2$ is a subspace of V.

Proof :

As $0 \in w_1 \cap w_2$, $w_1 \cap w_2$ is non-empty.

Consider $u, v \in w_1 \cap w_2, \alpha \in F$.

Then u, v $\in w_1, \alpha \in F$ and u,v $\in w_2, \alpha \in F$

$u + \alpha v \in w_1$ and $u + \alpha v \in w_2$

So, $u + \alpha v \in w_1 \cap w_2$

Hence $w_1 \cap w_2$ is a subspace of V.

PROBLEMS BASED ON SUBSPACES

1. Let $V = R^3$. The XY-plane $w_1 = \{(x,y,0) : x, y \in R\}$ and the XZ-plane $w_2 = \{(x,0,z) : x, z \in R\}$.These are subspace of R^3 . Then $w_1 \cap w_2 = \{(x,0,0) : x \in R\}$ is the x-axis.

Solution :

Let $v \in V$, $v = (x, y, z) \in V$

$v = (x, y, 0)+(0,0,z) \in w_1 + w_2$

So, $V \subseteq w_1 + w_2 \subseteq V$

Hence $V = w_1 + w_2$

2. Express the polynomial $3t^2 + 5t - 5$ as a linear combination of the polynomials $t^2 + 2t + 1, 2t^2 + 5t + 4, t^2 + 3t + 6$

Solution :

Let $a, b, c \in$ F such that

$$3t^2 + 5t - 5 = a(t^2 + 2t + 1) + b(2t^2 + 5t + 4) + c(t^2 + 3t + 6)$$

$$3t^2 + 5t - 5 = (a + 2b + c)t^2 + (2a + 5b + 3c)t + (a + 4b + 6c)$$

Comparing the co-efficients, we get

$$a + 2b + c = 3 \quad \ldots(1)$$

$$2a + 5b + 3c = 5 \quad \ldots(2)$$

$$a + 4\dot{v} + 6c = -5 \quad \ldots(3)$$

$(3) - (1) \Rightarrow \quad 2b + 5c = -8 \quad \ldots.(4)$

Multiply (1) by 2,

$$2a + 4b + 2c = 6 \quad \ldots\ldots\ldots(5)$$

(2) - (5) => $b + c = -1 \ldots$ (6)

Multiply (6) by 2,

$$2b + 2c = -2 \ \ldots (7)$$

$$(4) - (7) \Rightarrow 3c = -6$$

$$\therefore c = -2$$

Substituting c in (6),

$$b - 2 = -1$$

$$b = 2 - 1 = 1$$
$$\therefore b = 1$$

Substituting c, b in (1)

$$a + 2(1) - 2 = 3$$

$$a + 2 - 2 = 3$$

$$\Rightarrow a = 3$$

$$\therefore a = 3, b = 1, c = -2$$

Hence, $3t^2 + 5t - 5 = 3(t^2 + 2t + 1) + 1(2t^2 + 5t + 4)$

$$-2(t^2 + 3t + 6)$$

3. Let $V = R^3$, then which of the following sets is/are subspace(s) of V.

(i) $w_1 = \{(a, b, 0); a, b \in \mathbf{R}\}$

(ii) $w_2 = \{(a, b, 0); a \geq 0\}$

Solution :

(i) $\bar{0} = (0,0,0) \in w_1$, so $w_1 \neq \phi$

Let $v_1, v_2 \in w_1, \alpha \in$ R

Then, $v_1 = (a, b, 0)$ and $v_2 = (c, d, 0)$ for some $a, b, c, d \in \mathbb{R}$

$v_1 + v_2 = (a + c, b + d, 0) \in w_1$

$\alpha v_1 = (\alpha a, \alpha b, 0) \in w_1$

Hence w_1 is a subspace of V.

(ii) Consider $w_2 = \{(a, b, 0); a \geq 0\}$

Here we should take the value of a as zero or positive.

Let $V = (2,1,0) \in w_2$

But under scalar multiplication, the vector is not in w_2

That is $-v = (-2, -1,0) \notin w_2$

$(-1)v \notin w_2$

Hence w_2 is not a subspace of V

4. Let V be a vector space of all 2×2 matrices over real numbers. Determine whether W is a subspace of V or not, where

(i) W consists of all matrices with non-zero determinant.

(ii) W consists of all matrices A such that $A^2 = A$.

Solution :

(i) Let $w = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{W}, \mathrm{W}$ is a non-empty subset of V.

Consider $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}, B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix} \in W$ and $\alpha, \beta \in R$

$$\alpha A = \begin{bmatrix} \alpha x_1 & 0 \\ 0 & \alpha y_1 \end{bmatrix} \text{ and } \alpha B = \begin{bmatrix} \beta x_2 & 0 \\ 0 & \beta y_2 \end{bmatrix}$$

$$\alpha A + \beta B = \begin{bmatrix} \alpha x_1 + \beta x_1 & 0 \\ 0 & \alpha y_1 + \beta y_2 \end{bmatrix} \in \mathrm{W}$$

Hence W is a subspace of V.

(ii) W is not a subspace of V because w is not closed under addition.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so that

$$\mathrm{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \mathrm{A}$$

$\therefore A \in W$

But $A + A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \neq \mathrm{A} + \mathrm{A}$$

Thus $\mathrm{A} + \mathrm{A} \notin \mathrm{W}$

7. Let $\mathbf{V} = \left\{\mathbf{A}/\mathbf{A} = \left[a_{ij}\right]_{n\times n}, a_{ij} \in \mathbf{R}\right\}$ be a vector space over **R**. Show $W = \{\mathbf{A} \in \mathbf{V}/\mathbf{AX} = \mathbf{XA}$ for all $\mathbf{A} \in \mathbf{V}\}$ is a sub-space of $\mathbf{V}(\mathbf{R})$

Solution :

Since $0\mathrm{X} = 0 = \mathrm{X}0$ for all $\mathrm{X} \in \mathrm{V}$

$\Rightarrow 0 \in \mathrm{W}$. Thus W is non-empty.

Now, let $\alpha, \beta \in R$ and $A_1, A_2 \in W$

$\Rightarrow \mathrm{A}_1\mathrm{X} = \mathrm{X}\mathrm{A}_1$ and $\mathrm{A}_2\mathrm{X} = \mathrm{X}\mathrm{A}_2$ for all $\mathrm{X} \in \mathrm{V}$

$$\therefore (\alpha A_1 + \beta A_2)X = (\alpha A_1)X + (\beta A_2)X$$

$$= \alpha(A_1X) + \beta(A_2X)$$

$$= \alpha(\mathrm{XA}_1) + \beta(\mathrm{XA}_2)$$

$$= \mathrm{X}(\alpha\mathrm{A}_1) + \mathrm{X}(\beta\mathrm{A}_2)$$

$$= \mathrm{X}(\alpha\mathrm{A}_1 + \beta\mathrm{A}_2)$$

$$= \alpha\mathrm{A}_1 + \beta\mathrm{A}_2 \in \mathrm{W}$$

Hence W is a vector space of $V(R)$.

Theorem : 3. If S is any subset of a vector space V(F), then S is a subspace of V(F) if and only if L(S) = S.

Proof:

Given S is a subspace of $V(F)$

To prove $L(S) = S$

Let $x \in L(S) \Rightarrow$ there exists $x_1, \ldots, x_n \in S$

$\alpha_1, \alpha_2, \ldots, \alpha_n \in F$

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \in S$$

$L(S) \subset S$ ….. (1)

Also $S \subset L(S)$ ... (2) [Since S is a subspace of V(F)]

From (1) and (2), L(S) = S

Conversely, Given $L(S) = S$

To prove: S is a subspace of $V(F)$

Since $L(S)$ is a subspace of $V(F)$

∴ S is also a subspace of V(F)

8. Let V be the set of all solutions of the differential equation $2y'' - 7y' + 3y = 0$. Then V is a vector space over R.

Solution :

Let $f, g \in V$ and $\alpha \in R$.

Then $2f'' - 7f' + 3f = 0$ and

$$2g'' - 7g' + 3g = 0$$

$$2\frac{d^2}{dx^2}(f+g) - 7\frac{d}{dx}(f+g) + 3(f+g) = 0$$

Hence $f + g \in$ V

Also $2(\alpha f)^n - 7(\alpha f)' + 3(\alpha f) = 0$

Hence $\alpha f \in$ V

Hence V is a vector space over R.

9 Examine whether $(1, -3,5)$ belongs to the linear space generated by S, where $\mathrm{S} = \{(1,2,1), (1,1,-1), (4,5,-2)\}$ or not?

Solution :

Suppose $(1, -3,5)$ belongs to S.

∴ There exists scalars α, β, γ such that

$$(1, -3,5) = \alpha(1,2,1) + \beta(1,1,-1) + \gamma(4,5,-2)$$

$$(1, -3,5) = (\alpha + \beta + 4\gamma, 2\alpha + \beta + 5\gamma, \alpha - \beta - 2\gamma)$$

Comparing both sides, we get

$$\alpha + \beta + 4\gamma = 1 \quad \ldots\ldots\ldots(1)$$

$$2\alpha + \beta + 5\gamma = -3 \ \ldots\ldots\ldots(2)$$

$$\alpha - \beta - 2\gamma = 5\ldots\ldots(3)$$

Adding (1) and (3), we get

$$2\alpha + 2\gamma = 6 \Rightarrow \alpha + \gamma = 3 \ \ldots \ (4)$$

Adding (2) and (3), we get

$$3\alpha + 3\gamma = 2 \Rightarrow \alpha + \gamma = \frac{2}{3} \ldots \ (5)$$

Equation (4) and (5) are contradiction

Hence $(1, -3, 5)$ does not belong to linear space of S.

Remark :

The union of the subspace may not be a sub-space.