4.3 GENERAL WAVE BEHAVIOUR IN A CIRCULAR WAVEGUIDE (or) APPLICATION OF MAXWELL'S EQUATION FOR CIRCULAR WAVEGUIDE OR CYLINDRICAL WAVEGUIDE:

The general equations for field components is determined from Maxwell's curl equations.

$\nabla \times H = j\omega\, \varepsilon E$ …….(1)

$\nabla \times E = -j\omega\, \mu H$ …….(2)

Expanding equation (1),

$$\nabla \times H = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\emptyset} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\emptyset} & \frac{\partial}{\partial z} \\ H_\rho & \rho\, H_\emptyset & H_z \end{vmatrix} = j\omega\, \varepsilon\left[E_\rho\ \hat{\rho} + E_\emptyset\ \hat{\emptyset} + E_z\ \hat{z}\right]$$

Equating x, y, z components,

$$\frac{1}{\rho}\frac{\partial H_z}{\partial\emptyset} - \frac{\partial H_\emptyset}{\partial z} = j\omega\, \varepsilon\ E_\rho \quad \text{……(3a)}$$

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial\rho} = j\omega\, \varepsilon\, E_\emptyset \quad \text{………(3b)}$$

$$\frac{1}{\rho}\left[\frac{\partial(\rho\, H_\emptyset)}{\partial\rho} - \frac{\partial H_\rho}{\partial\emptyset}\right] = j\omega\, \varepsilon\, E_z \quad \text{……….(3c)}$$

Similarly

Expanding equation (2),

$$\nabla \times E = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\emptyset} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\emptyset} & \frac{\partial}{\partial z} \\ H_\rho & \rho\, H_\emptyset & H_z \end{vmatrix} = j\omega\, \mu\left[H_\rho\ \hat{\rho} + H_\emptyset\ \hat{\emptyset} + H_z\ \hat{z}\right]$$

Equating x, y, z components,

$\frac{1}{\rho}\frac{\partial E_z}{\partial \emptyset} - \frac{\partial E_\emptyset}{\partial z}$ = - jω ε H_ρ ……(3d)

$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}$ = jω ε $H_\emptyset$ ………(3e)

$\frac{1}{\rho}\left[\frac{\partial(\rho\, E_\emptyset)}{\partial \rho} - \frac{\partial E_\rho}{\partial \emptyset}\right] = -$ jω ε H_z ……….(3f)

$H_\emptyset = H_\emptyset^o\, e^{-\gamma z}$ …….(8)

Diff w.r.to 'z'

$\frac{\partial H_\emptyset}{\partial z} = H_\emptyset^o\, e^{-\gamma z}\,(-\gamma)$

$\frac{\partial H_\emptyset}{\partial z} = -\gamma\, H_\emptyset^o\, e^{-\gamma z}$

$\frac{\partial H_\emptyset}{\partial z} = -\gamma\, H_\emptyset$ ……..(9)

$\frac{\partial H_\rho}{\partial z} = -\gamma\, H_\rho$ ……….(10)

And also let,

$E_\emptyset = E_\emptyset^o\, e^{-\gamma z}$ …….(11)

Diff w.r.to 'z'

$\frac{\partial E_\emptyset}{\partial z} = E_\emptyset^o\, e^{-\gamma z}\,(-\gamma)$

$\frac{\partial E_\emptyset}{\partial z} = -\gamma\, E_\emptyset^o\, e^{-\gamma z}$

$\frac{\partial E_\emptyset}{\partial z} = -\gamma\, E_\emptyset$ ……..(12)

$\frac{\partial E_\rho}{\partial z} = -\gamma\, E_\rho$ ……….(13)

Sub the equ (9), (10), (12), (13) in equ (3),

$\frac{\partial H_z}{\rho \partial \emptyset} + \gamma\, H_\emptyset$ = jω ε E_ρ ………(14a)

$-\gamma\, H_\rho - \frac{\partial H_z}{\partial \rho}$ = jω ε $E_\emptyset$

$\gamma\, H_\rho + \frac{\partial H_z}{\partial \rho}$ = - jω ε $E_\emptyset$ ……..(14b)

$\frac{1}{\rho}\left[\frac{\partial(\rho\, H_\emptyset)}{\partial \rho} - \frac{\partial H_\rho}{\partial \emptyset}\right] =$ jω ε E_z ……..(14c)

$\frac{1}{\rho}\frac{\partial E_z}{\partial \emptyset} + \gamma\, E_\emptyset$ = - jω μ H_ρ …….(14d)

$\gamma\, E_\rho + \frac{\partial E_z}{\partial \rho\rho} = j\omega\mu\, H_\emptyset$ ………(14e)

$\frac{1}{\rho}\left[\frac{\partial(\rho\, E_\emptyset)}{\partial \rho} - \frac{\partial E}{\partial \emptyset}\right] = -j\omega\, \varepsilon\, H_z$ ……….(14f)

From (14b),

$\gamma\, H_\rho + \frac{\partial H_z}{\partial \rho} = - j\omega\, \varepsilon\, E_\emptyset$

$E_\emptyset = \frac{1}{-j\omega\, \varepsilon}\left[\gamma\, H_\rho + \frac{\partial H_z}{\partial \rho}\right]$

Sub the $E_\emptyset$ value in equ (14d),

From (14d),

$\frac{1}{\rho}\frac{\partial E_z}{\partial \emptyset} + \gamma\, E_\emptyset = - j\omega\, \mu\, H_\rho$

$\frac{1}{\rho}\frac{\partial E_z}{\partial \emptyset} + \gamma\left[\frac{1}{-j\omega\, \varepsilon}\left[\gamma\, H_\rho + \frac{\partial H_z}{\partial \rho}\right]\right] = - j\omega\, \mu\, H_\rho$

$H_\rho = \frac{j\omega\, \varepsilon}{\rho h^2}\frac{\partial E_z}{\partial \emptyset} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial \rho}$

$\gamma^2 + \omega^2\mu\, \varepsilon = h^2$

Similarly solving (14a) , (14d) we get,

$E_\rho = - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial \rho} - \frac{j\omega\mu}{\rho h^2}\frac{\partial H_z}{\partial \emptyset}$

$E_\emptyset = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial \rho} - \frac{\gamma}{\rho h^2}\frac{\partial E_z}{\partial \emptyset}$

$H_\emptyset = - \frac{\gamma}{\rho h^2}\frac{\partial H_z}{\partial \emptyset} - \frac{j\omega\mu}{h^2}\frac{\partial E_z}{\partial \rho}$

For TEM waves E_z & H_z = 0. All the field component vanish inside the cylindrical waveguide and hence no TEM wave can exists even in cylindrical or circular waveguides.

The wave equation in cylindrical coordinates is given by,

$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \emptyset^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu\, \varepsilon\, E_z$ ……(1)

$\frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial H_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \emptyset^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu\, \varepsilon\, H_z$ ……(2)

FIELD COMPONENTS OF TRANSVERSE MAGNETIC WAVES IN CIRCULAR WAVEGUIDE:

For TM waves, H_z = 0 and E_z is to be solved from eave equtions.

$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \emptyset^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu\, \varepsilon\, E_z$ ……(1)

$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \emptyset^2} + \gamma^2 E_z = -\omega^2 \mu\, \varepsilon\, E_z$

$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \emptyset^2} + (\gamma^2 + \omega^2 \mu\, \varepsilon)\, E_z = 0$

$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \emptyset^2} + h^2 E_z = 0$ ……(1)

$E_z = E_z^o\, e^{-\gamma z}$

${E_z}^o = P(\rho)\, Q(\emptyset)$ ……(2)

Where P is the function of ρ alone.

Q is the function $\emptyset$ alone.

Sub the value of E_z in equ (1),

$Q\frac{\partial^2 P}{\partial \rho^2} + \frac{P}{\rho^2}\frac{\partial^2 Q}{\partial \emptyset^2} + \frac{Q}{\rho}\frac{\partial P}{\partial \rho} + h^2 PQ = 0$ ……(3)

Dividing by PQ in equ (3),

$\frac{1}{P}\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{Q\rho^2}\frac{\partial^2 Q}{\partial \emptyset^2} + \frac{1}{P\rho}\frac{\partial P}{\partial \rho} + h^2 = 0$ …..(4)

Equ (4) i8s broken into two ordinary differential equations.

$\frac{1}{Q}\frac{\partial^2 Q}{\partial \emptyset^2} = -n^2$

$\frac{\partial^2 Q}{\partial \emptyset^2} = -n^2\, Q$ ……(5)

Sub the equ (5) in equ (4)

$\frac{1}{P}\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{Q\rho^2}(-n^2\, Q) + \frac{1}{P\rho}\frac{\partial P}{\partial \rho} + h^2 = 0$

$\frac{1}{P}\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{P\rho}\frac{\partial P}{\partial \rho} + h^2 - \frac{n^2}{\rho^2} = 0$

Multiply by P throughout

$\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial P}{\partial \rho} + P\left(h^2 - \frac{n^2}{\rho^2}\right) = 0$

Multiply by h^2 in the denominator

$\frac{\partial^2 P}{\partial (\rho h)^2} + \frac{1}{\rho h}\frac{\partial P}{\partial (\rho h)} + P\left(1 - \frac{n^2}{(\rho h)^2}\right) = 0$ …..(6)

This is a standard form of bessels equation interms of ρh. Using only the solution of first kind we get,

P(ρh) = J_n (ρh) ……(7)

The solution of $\frac{\partial^2 Q}{\partial \phi^2} = -n^2$ Q is given by,

Q = [$A_n \cos n\,\emptyset + B_n \sin n\,\emptyset$] …..(8)

Using equ (7) and (8) in equ (2) wer get,

E_z = PQ $e^{-\gamma z}$

$E_z = J_n(\rho h)\,[A_n \cos n\,\emptyset + B_n \sin n\,\emptyset]\,e^{-\gamma z}$

B_n can be put to zero, because A_n and B_n determines only the orientation of the field because both components are periodic.

$E_z = J_n(\rho h)\,A_n \cos n\,\emptyset\,e^{-\gamma z}$ ……(9)

Using general expression for field components

E_ρ = - $\frac{\gamma}{h^2}\frac{\partial E_z}{\partial \rho} - \frac{j\omega\mu}{\rho h^2}\frac{\partial H_z}{\partial \emptyset}$

For TM waves, $H_z = 0$, $\gamma = j\beta$

E_ρ = - $\frac{\gamma}{h^2}\frac{\partial E_z}{\partial \rho}$

E_ρ = - $\frac{j\beta}{h^2} J_n{}'(\rho h)\,A_n \cos n\,\emptyset\,e^{-j\beta z}$

$E_\emptyset = \frac{-j\beta}{\rho h^2}\frac{\partial E_z}{\partial \emptyset}$

$E_\emptyset = \frac{j\beta}{\rho h^2} J_n(\rho h)\,A_n \sin n\,\emptyset\;n\,e^{-j\beta z}$

$$H_\rho = \frac{j\omega\, \varepsilon}{\rho h^{\,2}} \frac{\partial E_z}{\partial \emptyset}$$

$$H_\rho = -\frac{j\omega\, \varepsilon}{\rho h^{\,2}} J_n\,(\rho h)\, A_n \sin n\,\emptyset\;\, n\, e^{-j\beta z}$$

$$H_\emptyset = -\,\frac{j\omega\mu}{h^{\,2}} \frac{\partial E_z}{\partial \rho}$$

$$H_\emptyset = -\,\frac{j\omega\mu}{h^{\,2}} J_n{}'\,(\rho h)\, A_n \cos n\,\emptyset\; e^{-j\beta z}$$

CHARACTERISTICS OF TE WAVES IN CIRCULAR WAVEGUIDE:

h_{mn} should be replaced as h'_{mn}

i) PROPAGATION CONSTANT:

$$\gamma = \sqrt{(h'_{mn})^{\,2} - \,\omega^2\mu\,\varepsilon}$$

$$h'_{mn} = \frac{(ha)'_{mn}}{a}$$

ii) PHASE SHIFT

$$\beta = \sqrt{\omega^2\mu\,\varepsilon - \,(h'_{mn})^{\,2}}$$

iii) CUT – OFF FREQUENCY

At cut – off frequency f_c , $\gamma = 0$

$$\omega^2\mu\,\varepsilon = (h'_{mn})^2$$

$$h'_{mn} = \frac{(ha)'_{mn}}{a}$$

$$\omega_c = \frac{1}{\sqrt{\mu\,\varepsilon}}\, h'_{mn}$$

$$f_c = \frac{1}{2\pi\sqrt{\mu\,\varepsilon}}\, h'_{mn}$$

$$f_c = \frac{(ha)'_{mn}}{2\pi a\sqrt{\mu\,\varepsilon}}$$

iv) PHASE VELOCITY

$$v = \frac{\omega}{\beta}$$

$$v = \frac{\omega}{\sqrt{\omega^2 \mu\, \varepsilon - (h'_{mn})^2}}$$

v) WAVELENGTH (or) CUT – OFF WAVELENGTH

$$\lambda_c = \frac{v}{f_c}$$

$$\lambda_c = \frac{v}{\frac{(ha)'_{mn}}{2\pi a\sqrt{\mu\, \varepsilon}}}$$

$$\lambda_c = \frac{2\pi a}{(ha)'_{mn}}$$

Where a represents the radius of circular guide.